Signal & Systems

Lecture 1: Introduction to Signals

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Template made by Panagiotis Papagiannopoulos

What is a signal?

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Examples:

- Voltage or current over time
- Audio waveform
- Temperature sensor readings
- Pixel intensity along an image line

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  - $\mathbb{Z}^N$ (discrete variables, e.g. sample index, pixel grid)
  - Combinations of continuous and discrete dimensions
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  - If $N > 1$ we call the signals *multidimensional*
- **Output space $\mathcal{O}$:**
  - Typically $\mathbb{R}^M$ or $\mathbb{C}^M$
  - If $M > 1$, we have *multichannel* / *multivariate* signals
  - Examples: $M = 1$ (grayscale), $M = 3$ (RGB), $M = 6$ (IMU)

Signal taxonomy: domain and codomain

- **Domain (input):** what the signal is indexed by
  - time, space, pixels, frequency, sample index, ...
- **Codomain (output):** what values the signal takes
  - real, complex, vector-valued, symbols, ...

Many "signal types" are just different choices of domain and codomain.

Domain (time) vs Codomain (amplitude)

	Continuous input (time)	Discrete input (time)
Continuous amplitude	**Analog** $x(t)$	**Discrete-time / sampled** $x[n]$
Discrete amplitude	**Quantized** $x_Q(t)$	**Digital** $x_D[n]$

Sampling discretizes time (input). Quantization discretizes amplitude (output).

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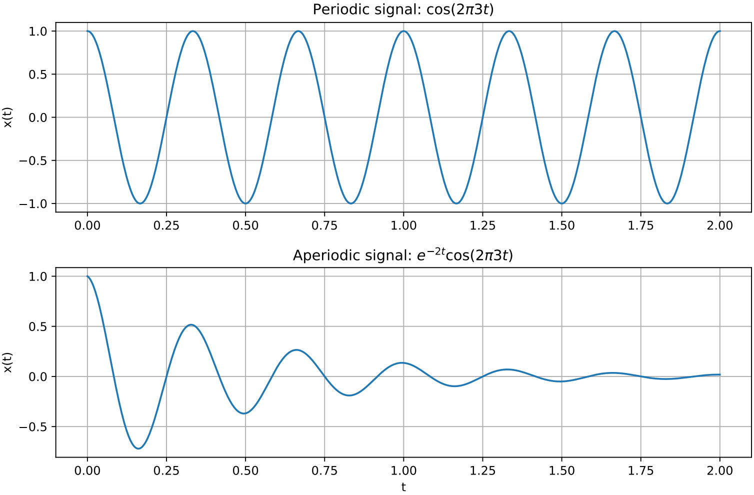
Example: $x(t) = \cos(2\pi f_0 t)$ has $T = 1/f_0$.

Examples: periodic vs aperiodic



Causal, anti-causal, two-sided (signal support)

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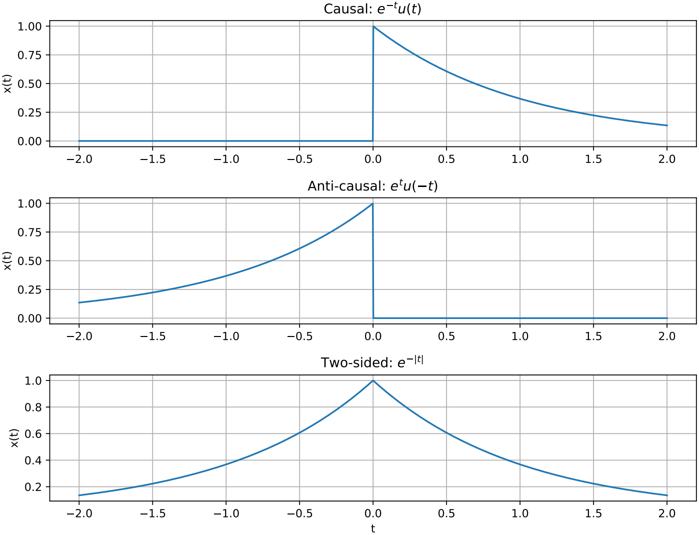
Examples: $e^{-t}u(t)$ (causal), $e^{t}u(-t)$ (anti-causal).


Causal: $e^{-t}u(t)$
x(t)
1.00
0.75
0.50
0.25
0.00
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
Anti-causal: $e^{t}u(-t)$
x(t)
Two-sided: $e^{-|t|}$
x(t)
1.0
0.8
0.6
0.4
0.2
t

Even and odd signals

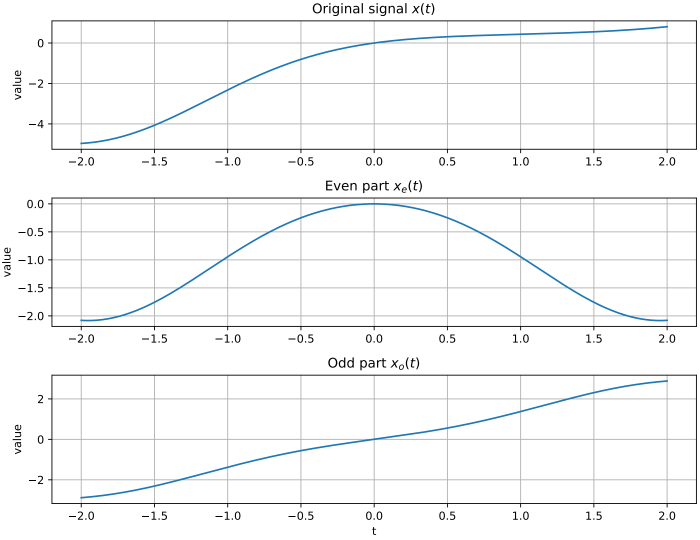
- **Even:** $x(t) = x(-t)$ **Odd:** $x(t) = -x(-t)$

Even and odd signals

- **Even:** $x(t) = x(-t)$ **Odd:** $x(t) = -x(-t)$
- Any signal can be decomposed:

$$
\begin{aligned}
x_e(t) &= \frac{1}{2}\big(x(t) + x(-t)\big),\\
x_o(t) &= \frac{1}{2}\big(x(t) - x(-t)\big),\\
x(t) &= x_e(t) + x_o(t).
\end{aligned}
$$

Examples: even/odd decomposition



Deterministic vs stochastic signals

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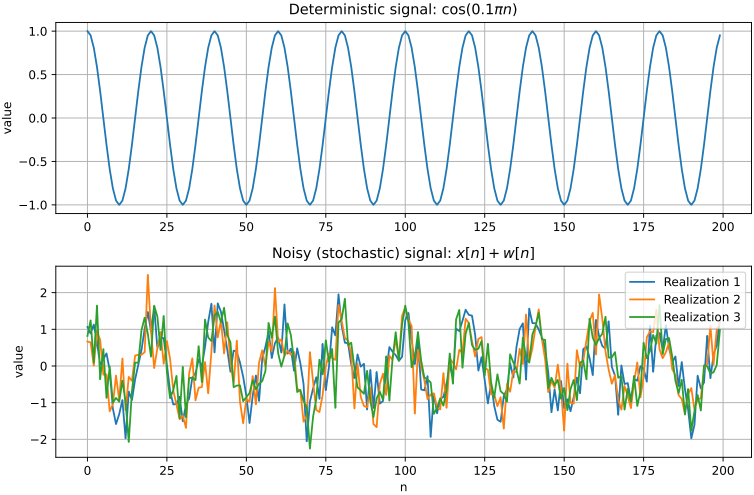
Example: sensor measurement $y[n] = x[n] + w[n]$ where $w[n]$ is noise.

Example: deterministic signal + noise



Signal Types in this course

In this course, we will mostly see:

- One dimensional signals (input $N = 1$);
- One output channel (output $M = 1$);
- Continuous-time (analog) or discrete-time signals;
- Deterministic signals (for most of the course);
- Stochastic signals (for the last part of the course).

Energy and power signals

- **Energy:**

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$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt, \qquad P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

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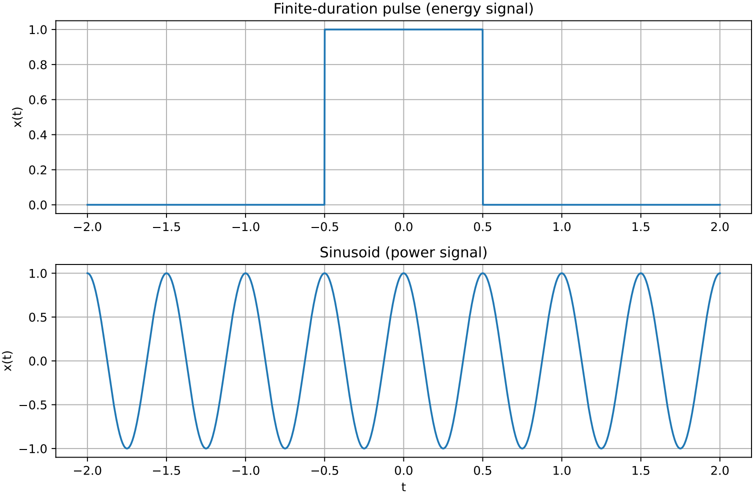
- **Energy signal:** $0 < E < \infty$ (and $P = 0$)

- **Power signal:** $0 < P < \infty$ (and $E = \infty$)

Examples: finite pulse $\Rightarrow$ energy, sinusoid $\Rightarrow$ power.

Examples: energy vs power signals



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- **Time shift:** move left/right
- **Time scaling (CT):** compress/expand in time
- **Time reversal:** flip around $t = 0$
- **Amplitude scaling:** multiply the signal values

We will use these constantly for system analysis and Fourier methods.

Time shift

- **Continuous time:**

$$y(t) = x(t - t_0)$$

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Interpretation: When $n_0/t_0 > 0$ the signal delays, and when $n_0/t_0 < 0$ the signal advances.

Time reversal

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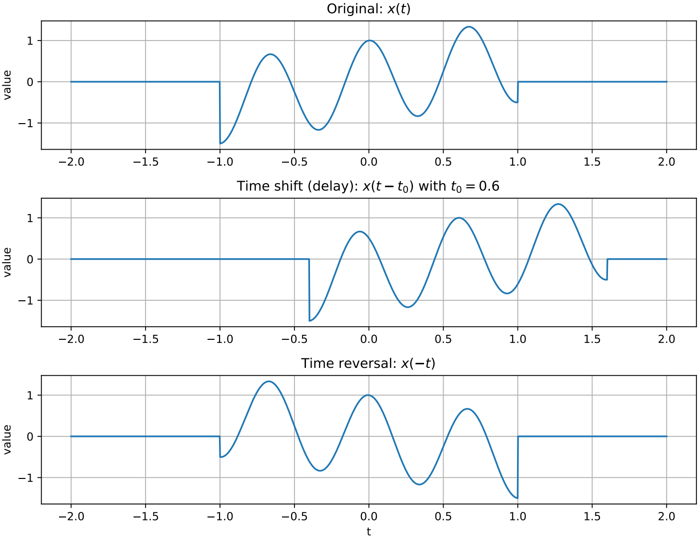
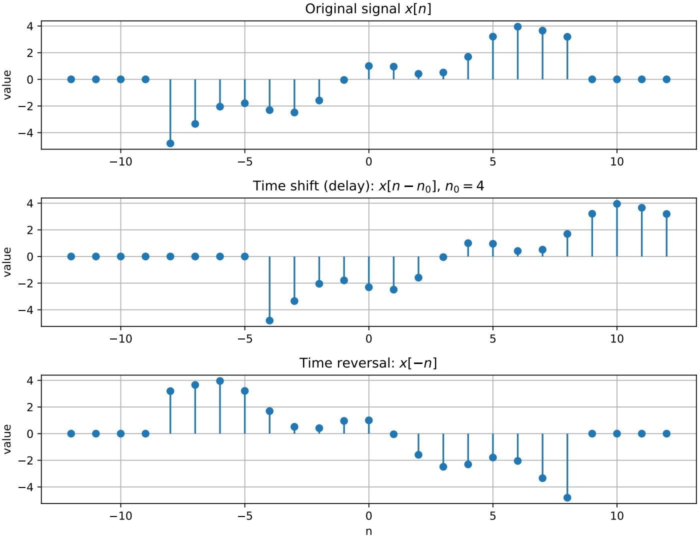
- **Discrete time:** $y[n] = x[-n]$

Interpretation: mirror the signal around $t = 0$ (or $n = 0$).

Examples: shift and reversal



Discrete-time examples: shift and reversal



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- $0 < |a| < 1$: **time expansion** (slower signal)
- $a < 0$: includes **reversal** as well (flip + scaling)

Discrete-time scaling is subtle: $x[an]$ is only defined on integer indices.

Amplitude scaling

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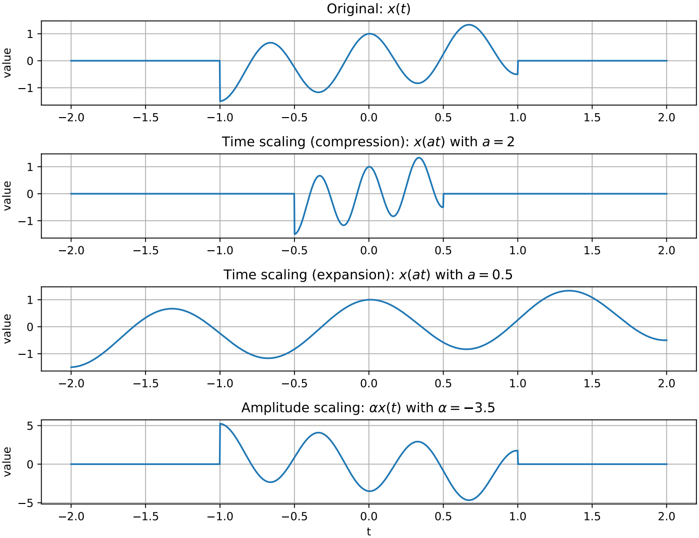
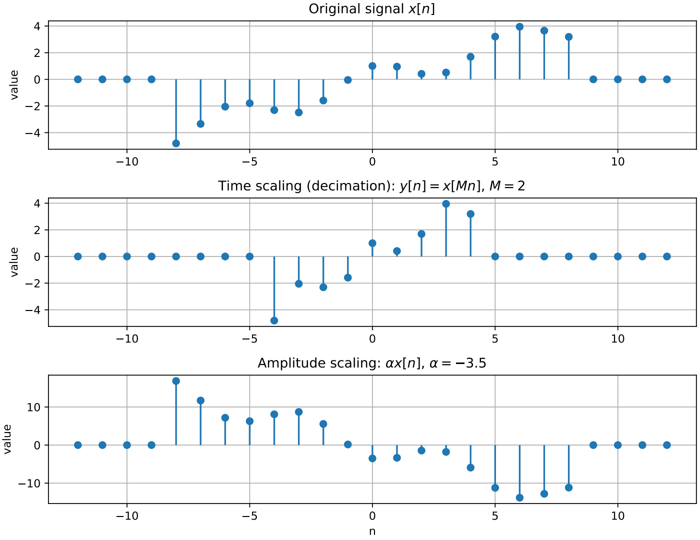
- $|\alpha| > 1$: amplify
- $0 < |\alpha| < 1$: attenuate
- $\alpha < 0$: sign inversion (vertical flip)

Examples: time scaling and amplitude scaling



Discrete-time examples: time and amplitude scaling



- **Unit step**

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CT - Common signals

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- **Ramp**

$$r(t) = t\,u(t)$$

CT - Common signals (2)

- **Sign function**

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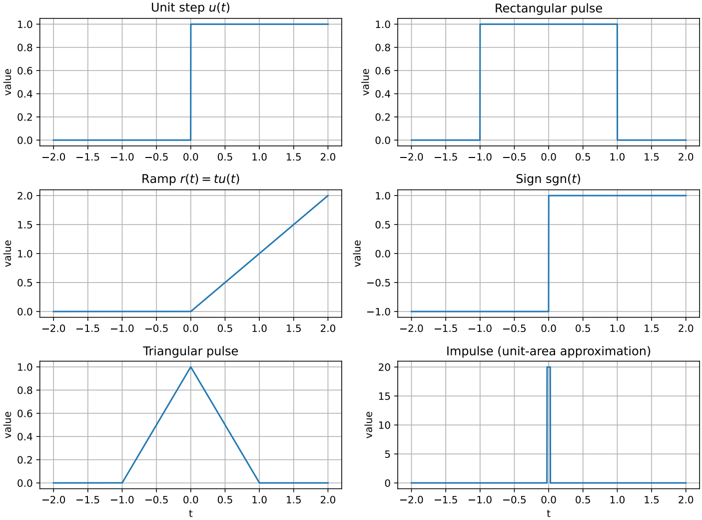
- **Impulse (Dirac)**

$$\delta(t), \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Common signals: continuous-time examples



Impulse and step: fundamental connection

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Interpretation:

- The impulse represents an *instantaneous change*
- The step is the *accumulation* of that change

Time shift and scaling of the impulse

- **Time shift:**

$$\delta(t - t_0)$$

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- Combined:

$$\delta(a(t - t_0)) = \frac{1}{|a|}\,\delta(t - t_0)$$

Key properties of the impulse

- **Unit area:**

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- **Shifting (sampling) property:**

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$$

- **Multiplication by a function:**

$$x(t)\, \delta(t - t_0) = x(t_0)\, \delta(t - t_0)$$

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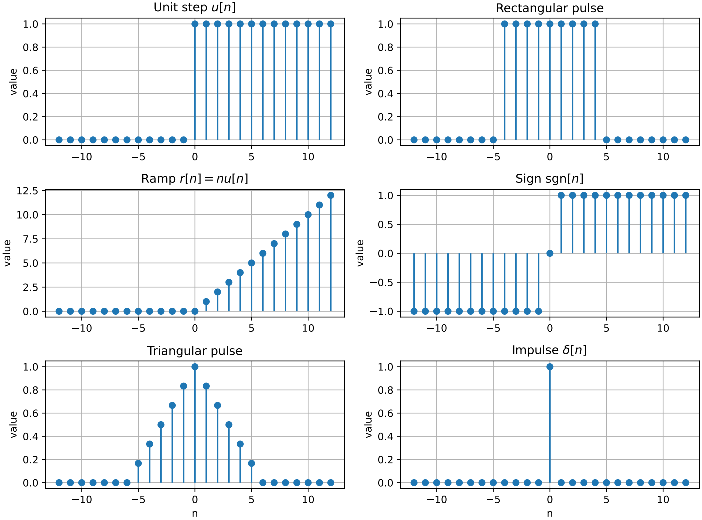
- **Impulse (Kronecker)**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Common signals: discrete-time examples



Discrete-time analogue of impulse-step connection

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- **Shifting property:**

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n - n_0] = x[n_0]$$

Complex exponential

- Continuous-time:

$$\begin{aligned} x(t) &= e^{(\sigma + j\omega)t} \\ &= e^{\sigma t} e^{j\omega t} \\ &\overset{\text{Euler's formula}}{=} e^{\sigma t}\big(\cos(\omega t) + j\sin(\omega t)\big) \end{aligned}$$

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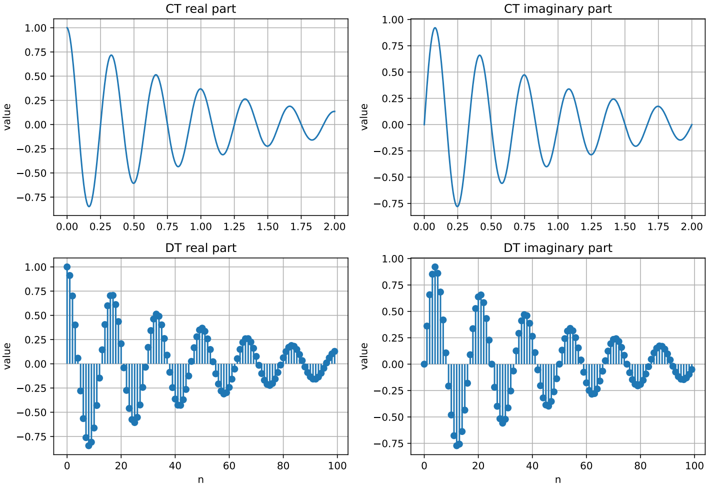
- **Euler's formula:** $e^{jx} = \cos(x) + j\sin(x)$.

Complex exponential: CT and DT examples



Energy and power under time shift (CT)

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Energy:

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |x(t - t_0)|^2 dt$$

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Matches the rule: $E_y = \frac{1}{|2|}E_x = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

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$x_1(t)$ periodic with period T_1, $\quad$ $x_2(t)$ periodic with period T_2

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In that case, a period is:

$$T = \mathrm{lcm}(T_1, T_2)$$

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$x(t)$ **is not periodic.**

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Sum is periodic if both are periodic and their periods have a common multiple.

Root-Mean-Square (RMS) value

- For a signal $x(t)$, the **RMS value** is defined as:

$$x_{\mathrm{rms}} = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}$$

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Key relation:

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Example:

$$x(t) = \alpha \cos(\omega t) \quad \Rightarrow \quad x_{\mathrm{rms}} = \frac{|\alpha|}{\sqrt{2}}$$

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RMS answers:

"How strong is the signal on average?"

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Why 20 and not 10? Because $P \propto A^2$.

dB rules of thumb and dynamic range

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Dynamic range: difference between the largest and smallest relevant levels:

$$\Delta_{\mathrm{dB}} = 20\log_{10}\left(\frac{A_{\max}}{A_{\min}}\right) \quad \text{or} \quad 10\log_{10}\left(\frac{P_{\max}}{P_{\min}}\right)$$

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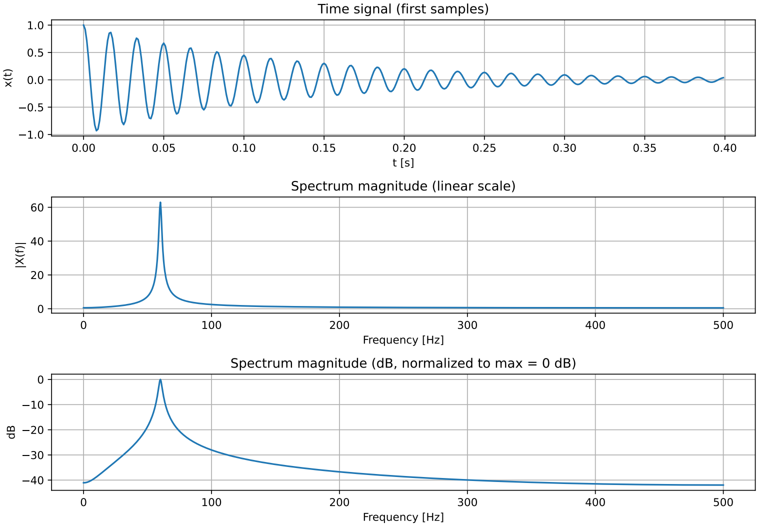
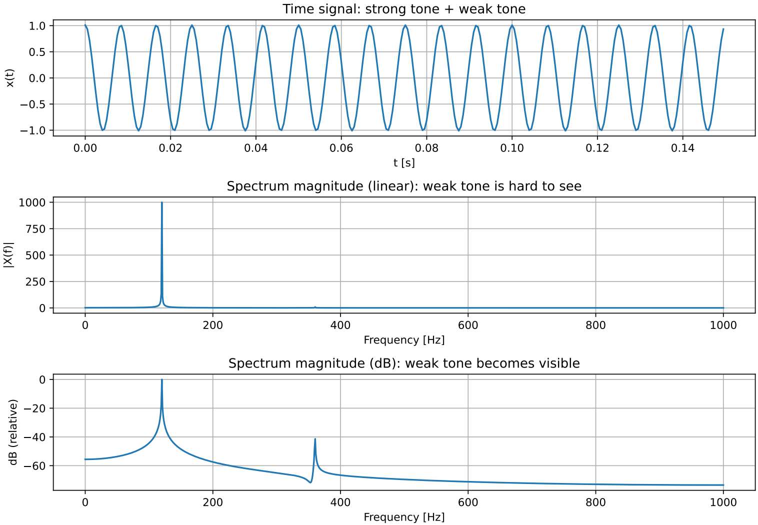
Often we plot **relative dB**:

$$20\log_{10}\left(\frac{m}{m_{\mathsf{max}}} + \varepsilon\right)$$

Example: linear vs dB magnitude



Example: dB reveals weak components



Thank you

- **Any Questions?**
- **Office Hours**:
  - **Mon & Tue** *(09:00-11:00)*
  - 24/7 by email (`costashatz@upatras.gr`, subject: *ECE_SS_AM*)
- **Material and Announcements**

Laboratory of Automation & Robotics