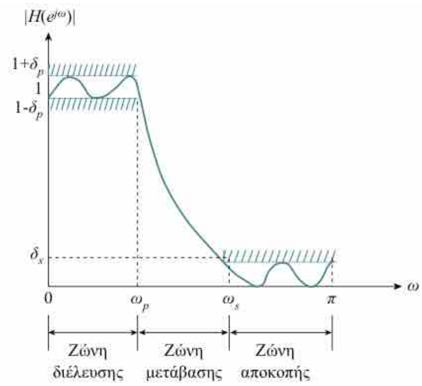
ΑΠΟΚΡΙΣΗ ΣΥΧΝΟΤΗΤΑΣ

- FREQUENCY RESPONSE -



FREQUENCY RESPONSE

Frequency Response of Discrete-Time Systems

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)$$

For an exponential input $x(n) = e^{j\omega n}$ $\quad -\infty < n < \infty$

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)e^{j\omega(n-m)} = \left(\sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m}\right)e^{j\omega n}$$

$$y(n) = H\left(e^{j\omega}\right)e^{j\omega n}$$ where $$H(e^{j\omega}) = \sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m}$$

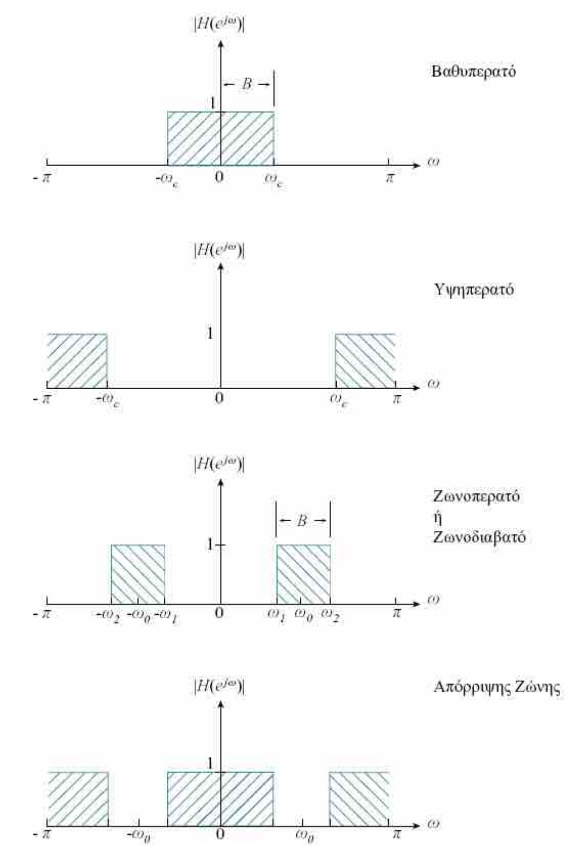
$H(e^{j\omega})$ is called freq. response of the system and is a complex function of ω with a period of 2π. It depends on the input freq. ω and the impulse response h(n).

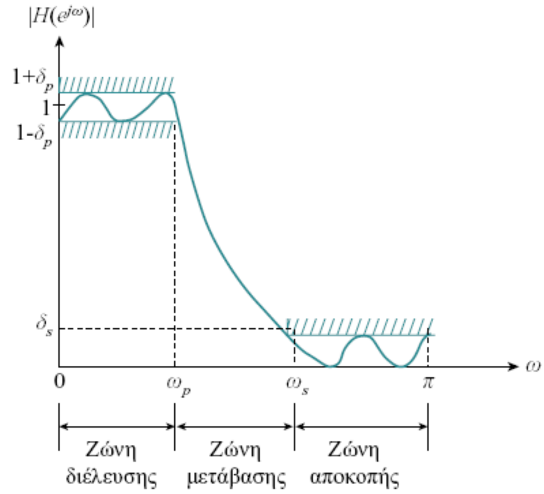
$$H(e^{j\omega}) = H_r(e^{j\omega}) + jH_i(e^{j\omega}) = \left|H(e^{j\omega})\right|e^{j\Theta(\omega)}$$

$$\Theta(\omega) = \arg\{H(e^{j\omega})\} = \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

(Magnitude & Phase response)

Filters





GRAPHICAL CONCEPTS

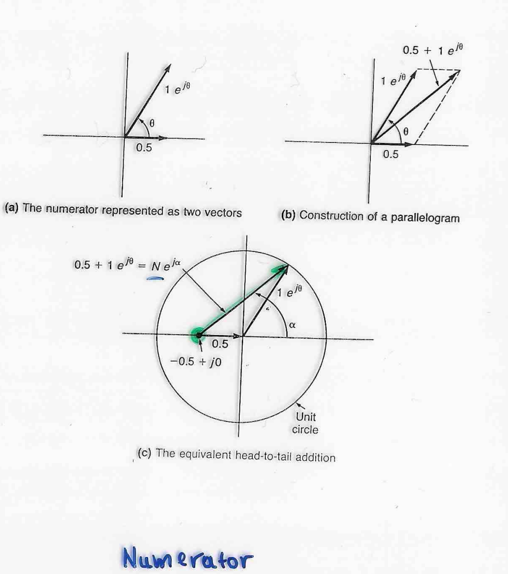
We start with the frequency response function

$$H(e^{j\theta}) = \frac{1 + 0.5e^{-j\theta}}{1 - 0.5e^{-j\theta}} = \frac{e^{j\theta} + 0.5}{e^{j\theta} - 0.5} = \frac{z+0.5}{z-0.5}$$

or

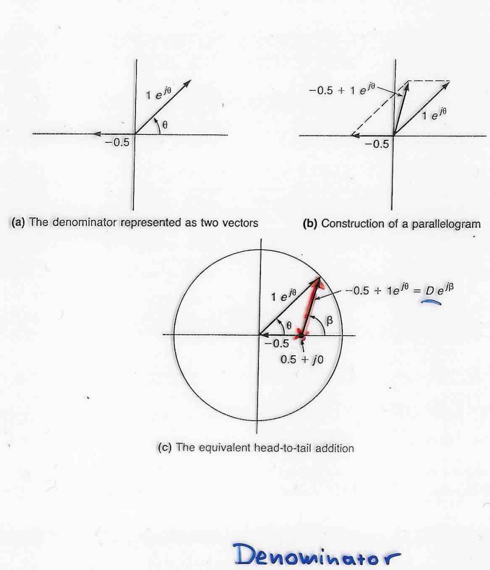
$$H(e^{j\theta}) = \frac{Ne^{j\alpha}}{De^{j\beta}}$$

$$H(e^{j\theta}) = \frac{N}{D}e^{j(\alpha-\beta)} = Me^{jP}$$



(a) The numerator represented as two vectors (b) Construction of a parallelogram

(c) The equivalent head-to-tail addition



(a) The denominator represented as two vectors (b) Construction of a parallelogram

(c) The equivalent head-to-tail addition

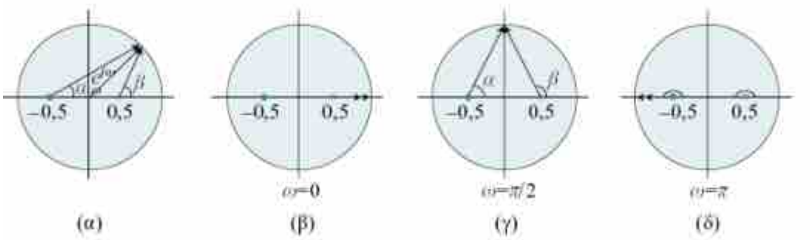
FREQUENCY RESPONSE

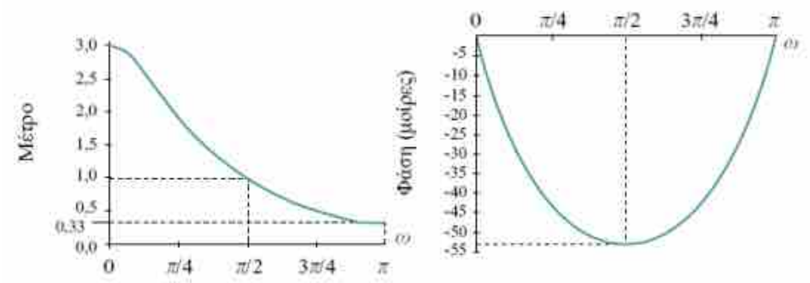
Geometrical (Graphical) Computation of the Frequency Response

Παράδειγμα 3.9

$H(z) = (z + 0{,}5)/(z - 0{,}5)$ με $|z| > 0{,}5$

$$H(e^{j\omega}) = \frac{0{,}5 + e^{j\omega}}{-0{,}5 + e^{j\omega}} = \frac{Ce^{j\alpha}}{De^{j\beta}} = Ve^{j\Theta}$$

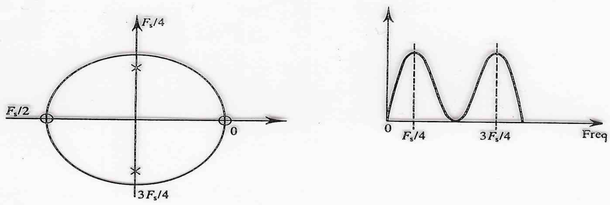




FREQUENCY RESPONSE

Pole-zero placement method

- When a zero is placed at a given point on the z-plane, the freq. response will be zero at the corresponding point. A pole on the other hand produces a peak at the corresponding freq. point



- Poles that are close to the unit circle give rise to large peaks, whereas zeros close to or on the unit circle produce troughs or minima.

- Notice that for coefficients of the filter to be real, the poles and zeros must either be real (that is lie on the positive or negative real axes) or occur in complex conjugate pairs.

FREQUENCY RESPONSE

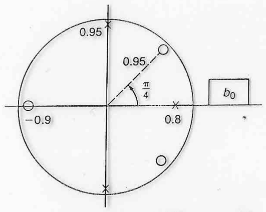
Effects of Poles and Zeros on the Frequency Response

- A filter pole close to the unit circle produces a large gain at nearby frequencies.

- A filter zero close to the unit circle produces a small gain at nearby frequencies.

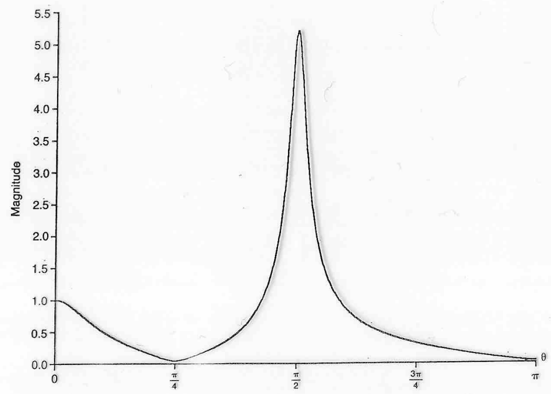
- The gain scale of the filter may be controlled by the factor b_0.

- "How about the phase?"
  Except in the simplest cases, it is difficult to learn much about phase from these diagrams.



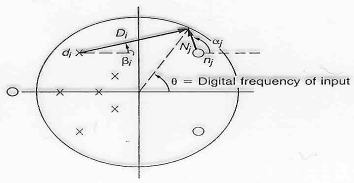
$$H(z) = b_0 \frac{(z+0.9)(z-0.95e^{j\pi/4})(z-0.95e^{-j\pi/4})}{(z-0.8)(z-0.95e^{j\pi/2})(z-0.95e^{-j\pi/2})}$$

(a) Pole-zero description



(b) Frequency response magnitude, *M*

GRAPHICAL DESIGN OF FILTERS



- For any causal (recursive or nonrecursive) **stable filter** with real coefficients, all of the poles of the generalized frequency response function $H(z)$ **must lie inside the unit circle.**

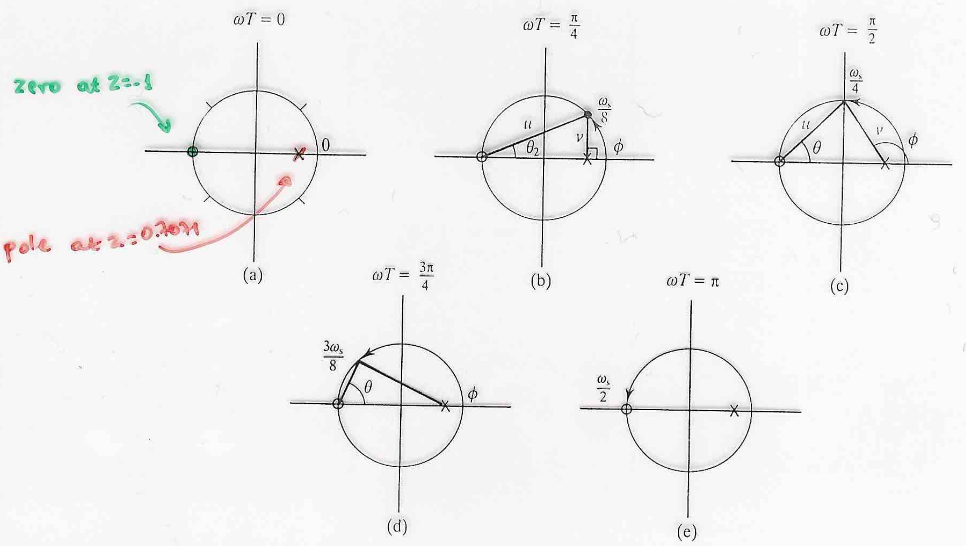
- The zeros may be located anywhere.

- Both poles and zeros may be real or complex, but complex poles or zeros must occur in conjugate pairs.

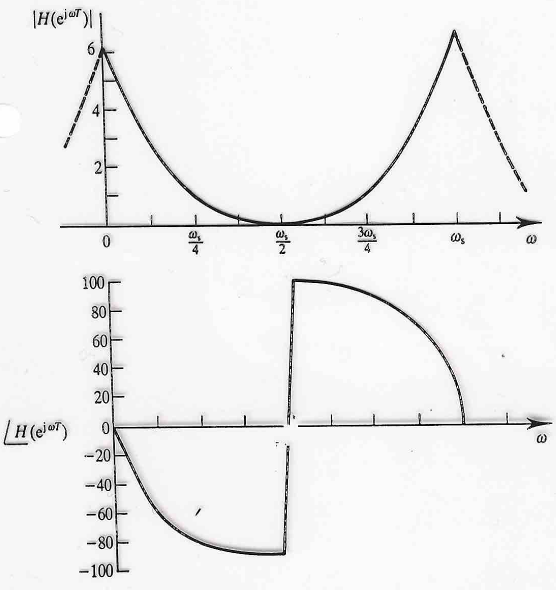
FREQUENCY RESPONSE

Frequency response estimation using geometric method and the pole–zero diagram

$$H(z) = \frac{z+1}{z-0.7071}$$



FREQUENCY RESPONSE



Remarks

1. The magnitude response is symmetrical and the phase response is antisymmetrical about half the sampling frequency. This is always the case when the coefficients a_k, b_k are real.

2. The freq. response of such systems is periodic with period ω_s, being consistent with the sampling theorem.

FREQUENCY RESPONSE

Example 5.1. This example illustrates the graphical estimation of the frequency response of two first-order systems.

The general form of the unit sample response for a given causal, first-order filter is $h[n] = a^n u[n]$ with $|a| < 1$ to ensure stability. Use the graphical method just discussed to estimate the frequency response for two different values of a, say ± 0.5.

Solution: First, for $a = +0.5$ we have

$$H(e^{j\theta}) = \sum_{m=0}^{m=\infty} (0.5)^m e^{-j\theta m} \quad (5.4)$$

where this infinite geometric sum can be written as

$$H(e^{j\theta}) = \frac{1}{1 - 0.5e^{-j\theta}}, \quad \text{since} \quad |0.5e^{-j\theta}| < 1$$

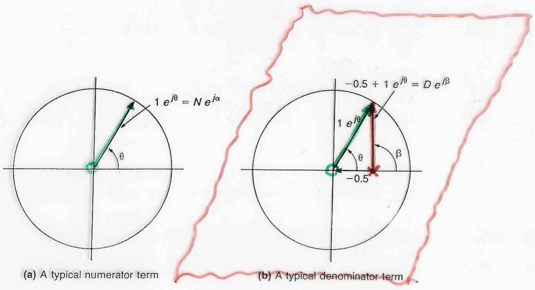
$$= \frac{e^{j\theta}}{e^{j\theta} - 0.5} = \frac{z}{z - 0.5} \quad (5.5)$$

← zero at z=0
← pole at z=0.5

This is of the general form

$$H(e^{j\theta}) = \frac{Ne^{j\alpha}}{De^{j\beta}} = Me^{jP}. \quad (5.6)$$

The numerator vector is $Ne^{j\alpha} = 1e^{j\theta}$ for all values of θ as in Fig. 5.5(a) and the denominator vector $De^{j\beta}$ starts with $0.5e^{j0°}$ at $\theta = 0$ and increases in length and in phase to $1.5e^{j180°}$ at $\theta = \pi$. Three easy-to-make computations are tabulated below and we note that M decreases monotonically as θ increases from 0 to π while D increases monotonically as shown in Fig. 5.5(b), and that the phase will always be negative because β is always greater than α for $0 < \theta < \pi$. Again this is a lowpass, negative phase (or phase lag) filter where the computer-generated frequency response is as shown in Fig. 5.6.



(a) A typical numerator term

(b) A typical denominator term

Frequency θ	Numerator $Ne^{j\alpha}$	Denominator $De^{j\beta}$	Ratio Me^{jP}
0	$1e^{j0°}$	$0.5e^{j0°}$	$2e^{j0°}$
$\pi/2$	$1e^{j90°}$	$1.12e^{j116.6°}$	$0.89e^{-j26.6°}$
π	$1e^{j180°}$	$1.5e^{j180°}$	$0.67e^{j0°}$

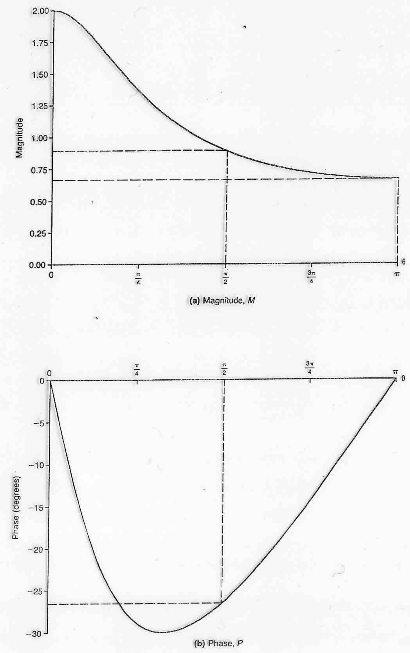


FIGURE 5.6 Lowpass filter frequency response

FREQUENCY RESPONSE

Now for the negative value of $a = -0.5$, the frequency response is

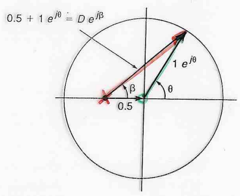
$$H(e^{j\theta}) = \frac{1}{1 - ae^{-j\theta}} = \frac{1}{1 + 0.5e^{-j\theta}}$$

$$= \frac{e^{j\theta}}{e^{j\theta} + 0.5} = \frac{z}{z + 0.5}$$

← zero at z=0
← pole at z=−0.5

$$= \frac{Ne^{j\alpha}}{De^{j\beta}} = Me^{jP} \quad (5.7)$$

and shown in Fig. 5.7 is the vector addition that results in the denominator term $De^{j\beta} = 0.5 + 1e^{j\theta}$. The numerator is always $1e^{j\theta}$ which leads to the quick tabulation indicated and we note that M increases monotonically (D decreases monotonically with θ). Here, however, the phase is positive because $\alpha = \theta$ is always greater than β. Figure 5.8 shows the amplitude and phase characteristics of this highpass filter that would be used to filter out sequences composed of low frequency sinusoids and pass more easily those containing higher frequencies.



Frequency θ	Numerator $Ne^{j\alpha}$	Denominator $De^{j\beta}$	Ratio Me^{jP}
0	$1e^{j0°}$	$1.5e^{j0°}$	$0.67e^{j0°}$
$\pi/2$	$1e^{j90°}$	$1.12e^{j63.4°}$	$0.89e^{j26.6°}$
π	$1e^{j180°}$	$0.5e^{j180°}$	$2e^{j0°}$

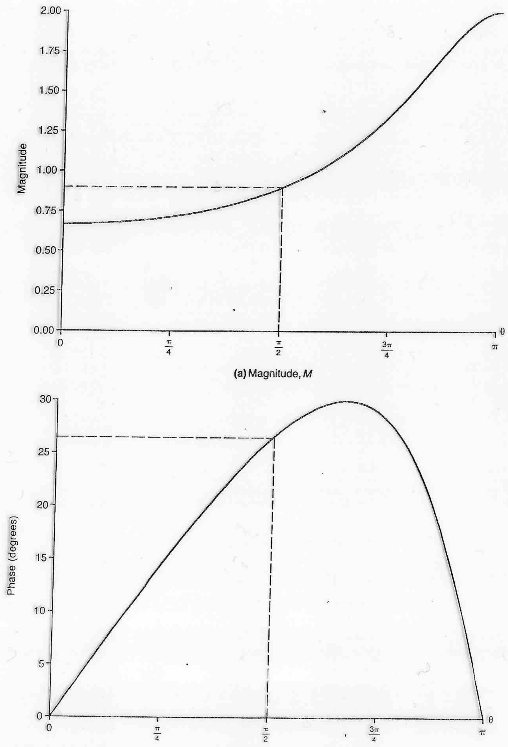


FIGURE 5.8 Frequency response for system with $H(e^{j\theta}) = e^{j\theta}/(e^{j\theta} - 0.5)$

FREQUENCY RESPONSE

Comment: To reflect for a moment on this example, we note the simplicity inherent in digital filters. The difference equation that corresponds to the unit sample response of $h(n) = a^n u(n)$ is

$$y(n) = ay(n-1) + x(n) \quad (5.8)$$

and the system diagram for this first-order recursive filter is given in Fig. 5.9. So, merely changing the sign of the filter coefficient a from + to − dramatically alters the filter characteristic: a lowpass becomes a highpass.

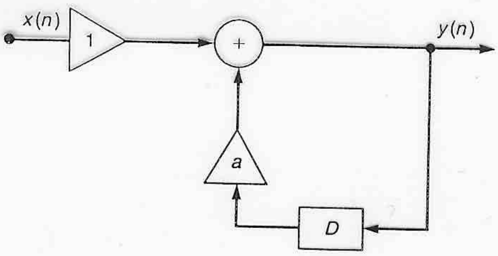


FIGURE 5.9 An implementation or realization of a first-order recursive filter

FREQUENCY RESPONSE

ΑΣΚΗΣΗ

Ένα διακριτού χρόνου σύστημα περιγράφεται από την εξίσωση διαφορών:

$$y(n) = \tfrac{1}{2}\, y(n-1) + 2\, x(n)$$

Α. Να υπολογιστεί η **συνάρτηση μεταφοράς** του συστήματος.

Β. Να *υπολογιστεί* και να *σχεδιαστεί* η **κρουστική απόκριση** του συστήματος.

Γ. Να *υπολογιστεί* (με γραφικό τρόπο) και να *σχεδιαστεί* η **απόκριση συχνότητας** του συστήματος.

FREQUENCY RESPONSE

Λύση

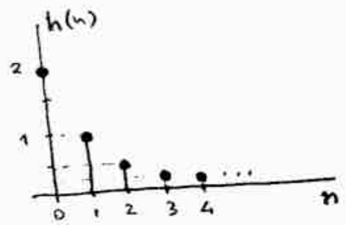
Α. Εφαρμόζοντας τον ΜΖ και στα δύο μέλη της εξίσωσης έχουμε:

$$Z\{y(n)\} = \frac{1}{2} Z\{y(n-1)\} + 2\, Z\{x(n)\} \Rightarrow$$

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2\, X(z) \Rightarrow \frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}} \Rightarrow H(z) = 2\cdot\frac{1}{1-\frac{1}{2}z^{-1}}$$

Β. $H(z) = 2\cdot\frac{1}{1-\frac{1}{2}z^{-1}} \xleftrightarrow{z^{-1}} h(n) = 2\cdot\left(\frac{1}{2}\right)^n u(n)$

Το σύστημα έχει έναν πόλο στο $z=\frac{1}{2}$ και ένα μηδενικό στο $z=0$.



FREQUENCY RESPONSE

Γ. Όπως είδαμε, το σύστημα έχει έναν πόλο στο $z=\frac{1}{2}$ και ένα μηδενικό στο $z=0$.

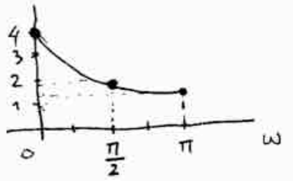
$\omega=0$ $\omega=\frac{\pi}{2}$ $\omega=\pi$

$$\left|H(e^{j\pi})\right| = 2\frac{1}{\frac{3}{2}} = 2\frac{2}{3} = 2\cdot 0.67 = 1.33$$

$$\left|H(e^{j0})\right| = 2\frac{1}{1/2} = 4$$

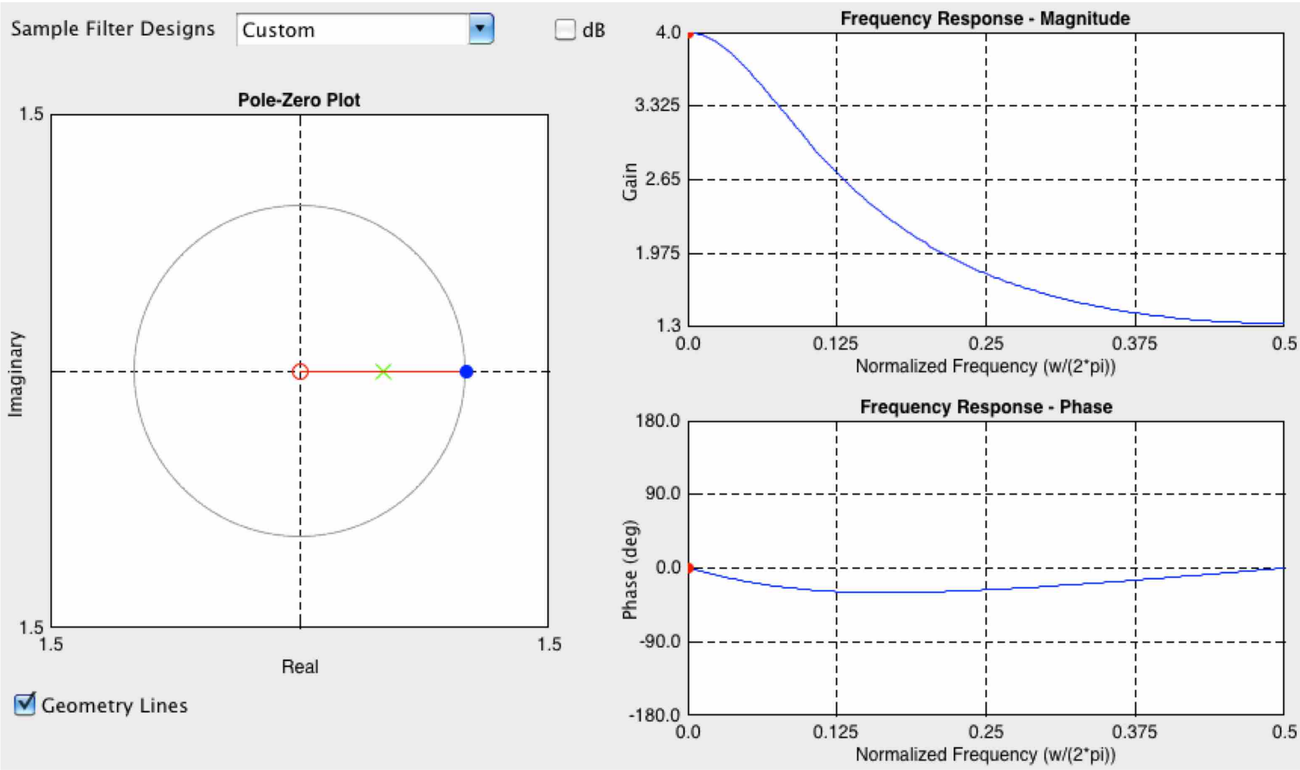
$$\left|H(e^{j\frac{\pi}{2}})\right| = 2\frac{1}{\sqrt{1+\frac{1}{4}}} = 2\frac{1}{\frac{\sqrt{5}}{2}} = 2\frac{2}{\sqrt{5}} = 2\frac{2}{2.24} = 2\cdot 0.89 = 1.79$$

$\left|H(e^{j\omega})\right|$



Άρα πρόκειται για βαθυπερατό (low-pass) φίλτρο.

Example 1: H(z) = 2z / (z-1/2)



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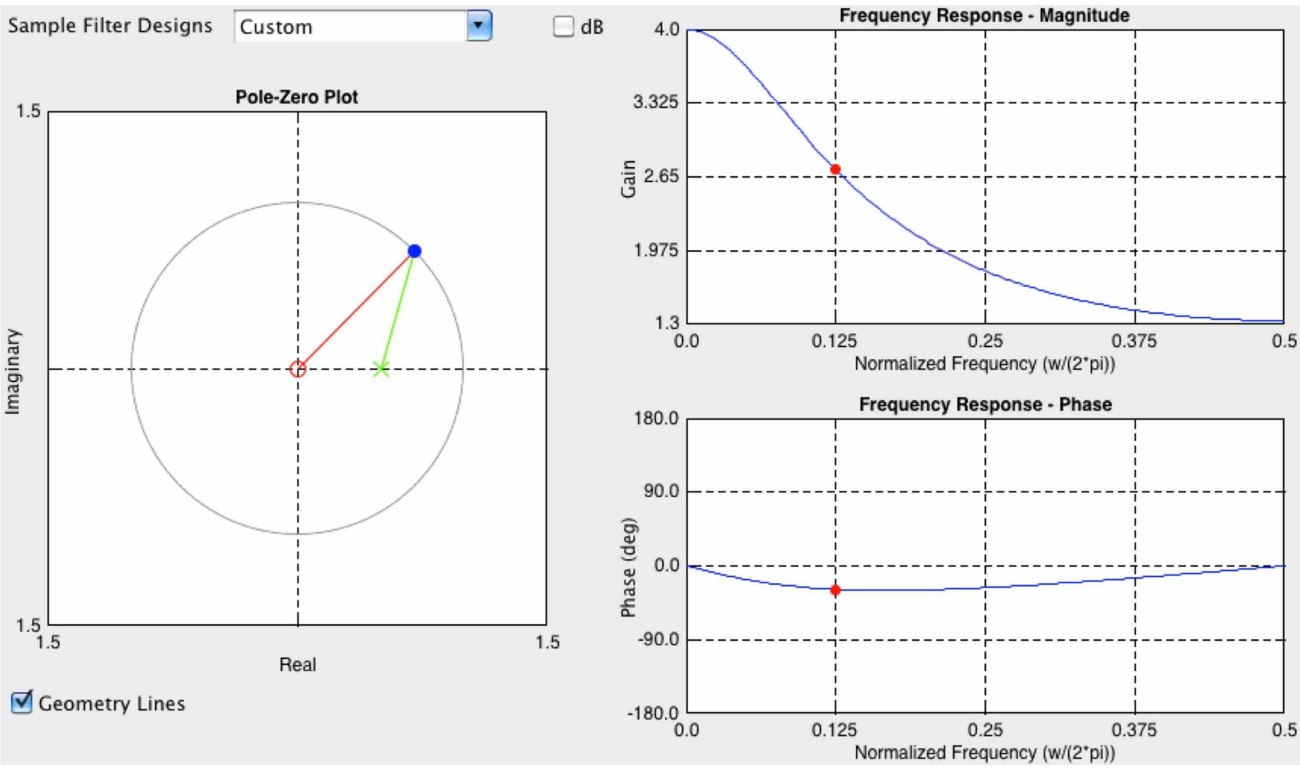

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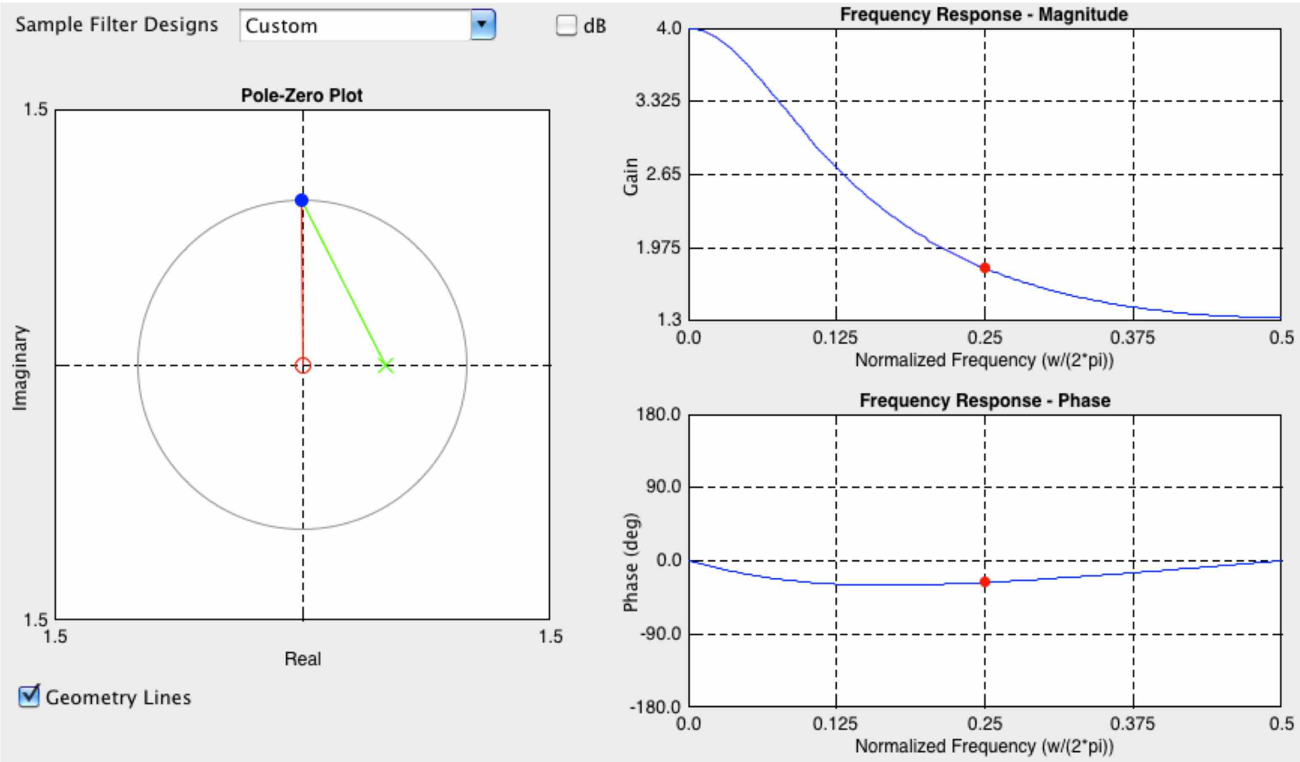

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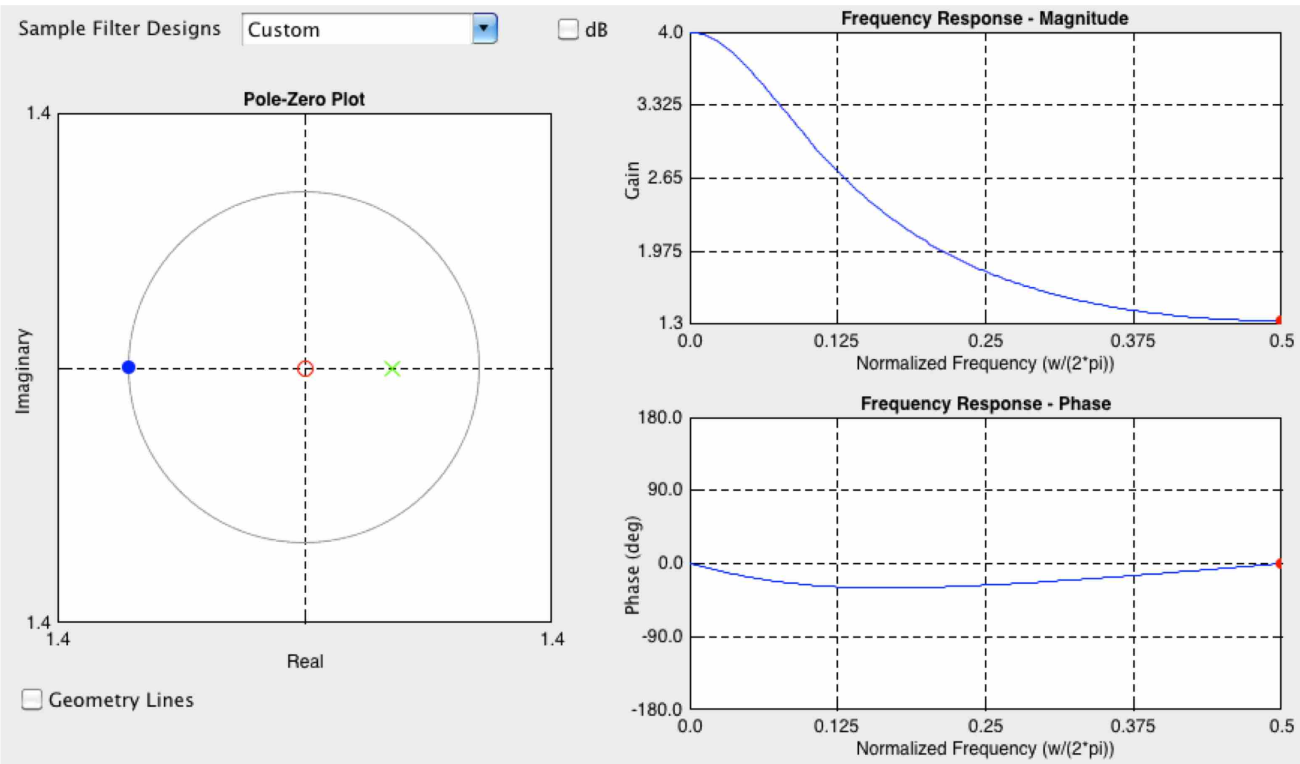
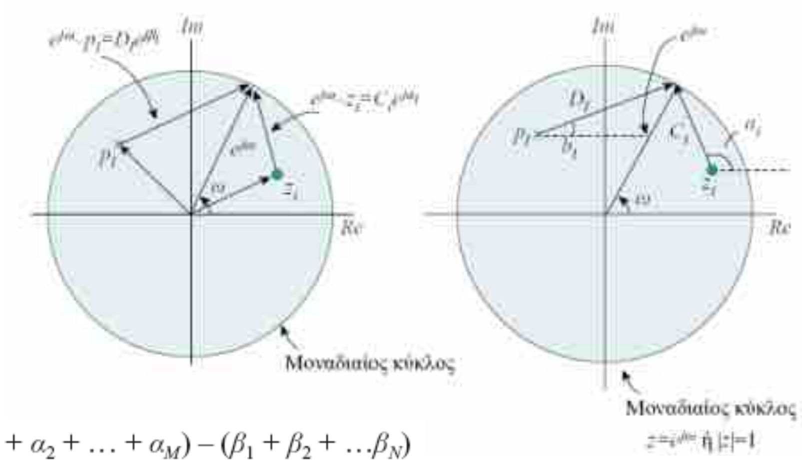


Pole-Zero Placement

$$H(z)=\frac{Y(z)}{X(z)} \quad \Longleftrightarrow \quad H(z)=\frac{b_0 z^{N-M}(z-z_1)(z-z_2)...(z-z_M)}{(z-p_1)(z-p_2)...(z-p_N)}$$

$$H(e^{j\omega})=b_0 e^{j\omega(N-M)}\frac{(e^{j\omega}-z_1)(e^{j\omega}-z_2)...(e^{j\omega}-z_M)}{(e^{j\omega}-p_1)(e^{j\omega}-p_2)...(e^{j\omega}-p_N)}$$

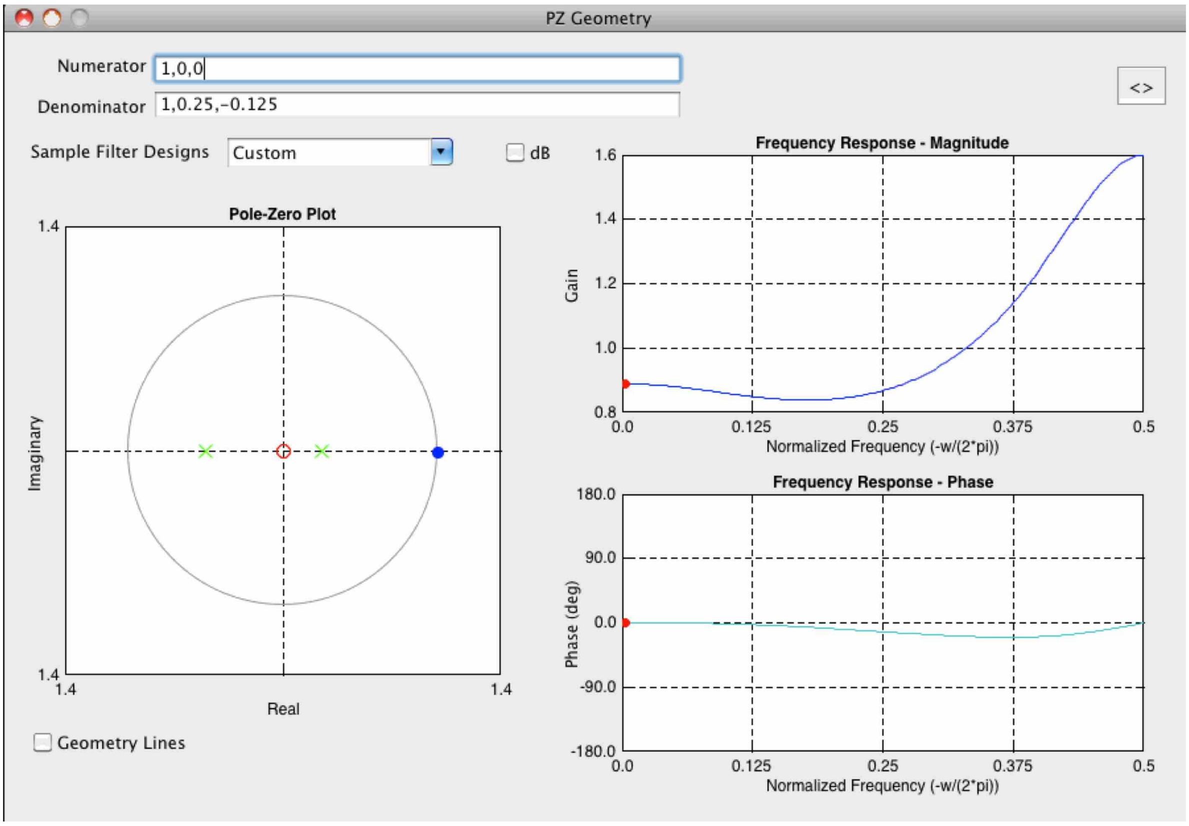
$$H(e^{j\omega})=b_0 e^{j\omega(N-M)}\frac{(C_1e^{j\alpha_1})(C_2e^{j\alpha_2})...(C_Me^{j\alpha_M})}{(D_1e^{j\beta_1})(D_2e^{j\beta_2})...(D_Ne^{j\beta_N})}\equiv Ve^{j\Theta}$$



$$V=|b_0|\frac{C_1C_2...C_M}{D_1D_2...D_N}$$

$$\Theta=\angle b_0+(N-M)\omega+(\alpha_1+\alpha_2+\ldots+\alpha_M)-(\beta_1+\beta_2+\ldots\beta_N)$$

Example 2: H(z) = z^2 / [(z-1/4) (z+1/2)]



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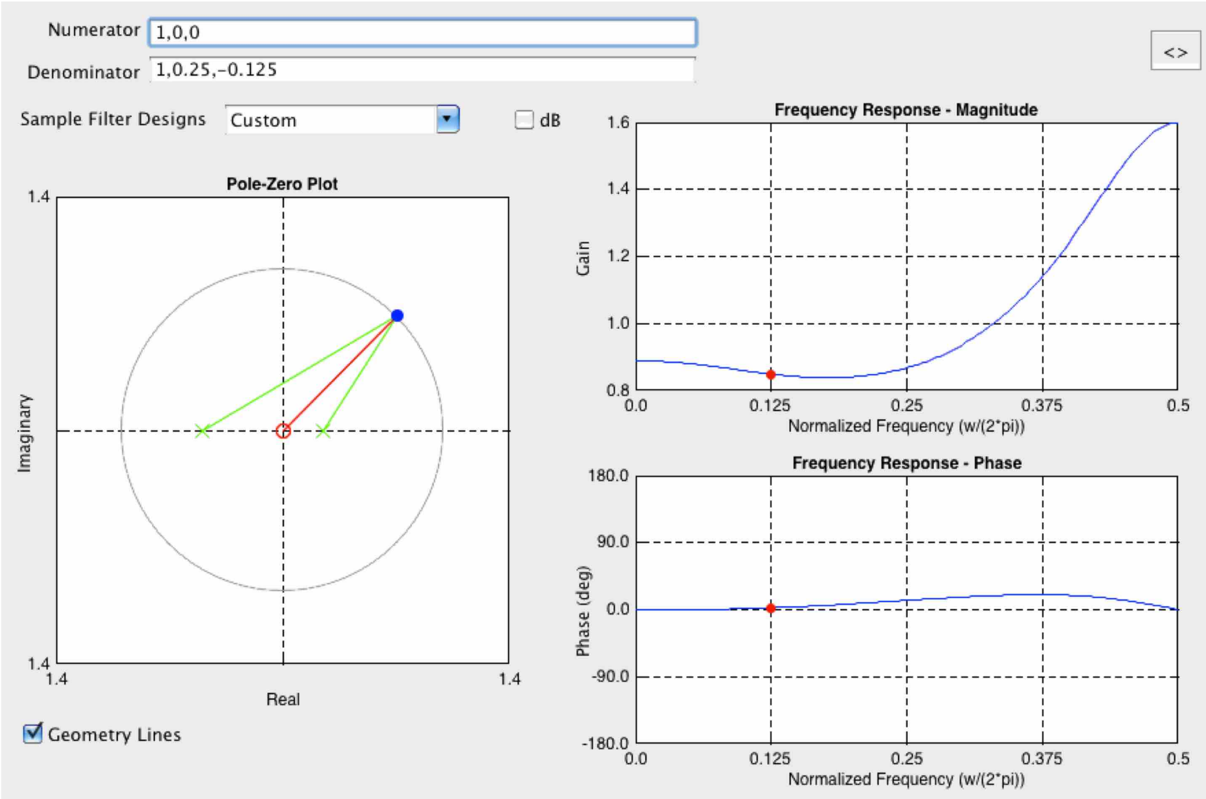

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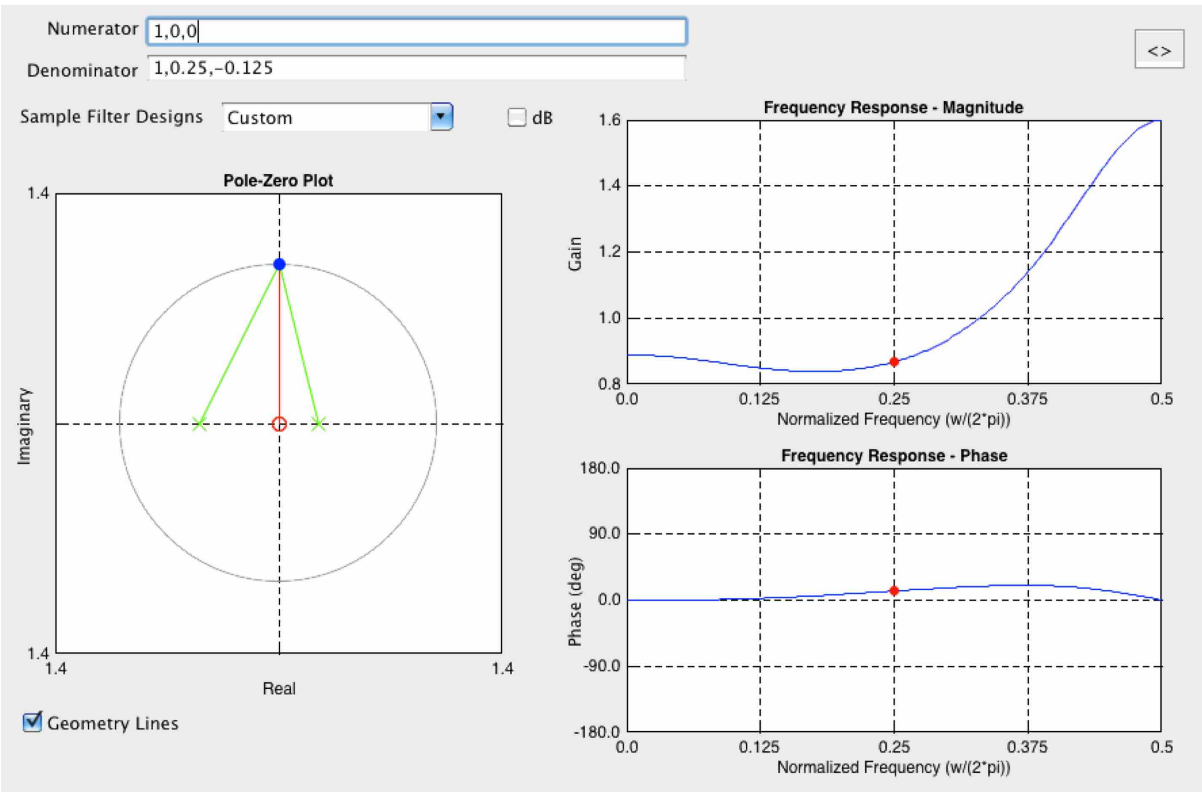
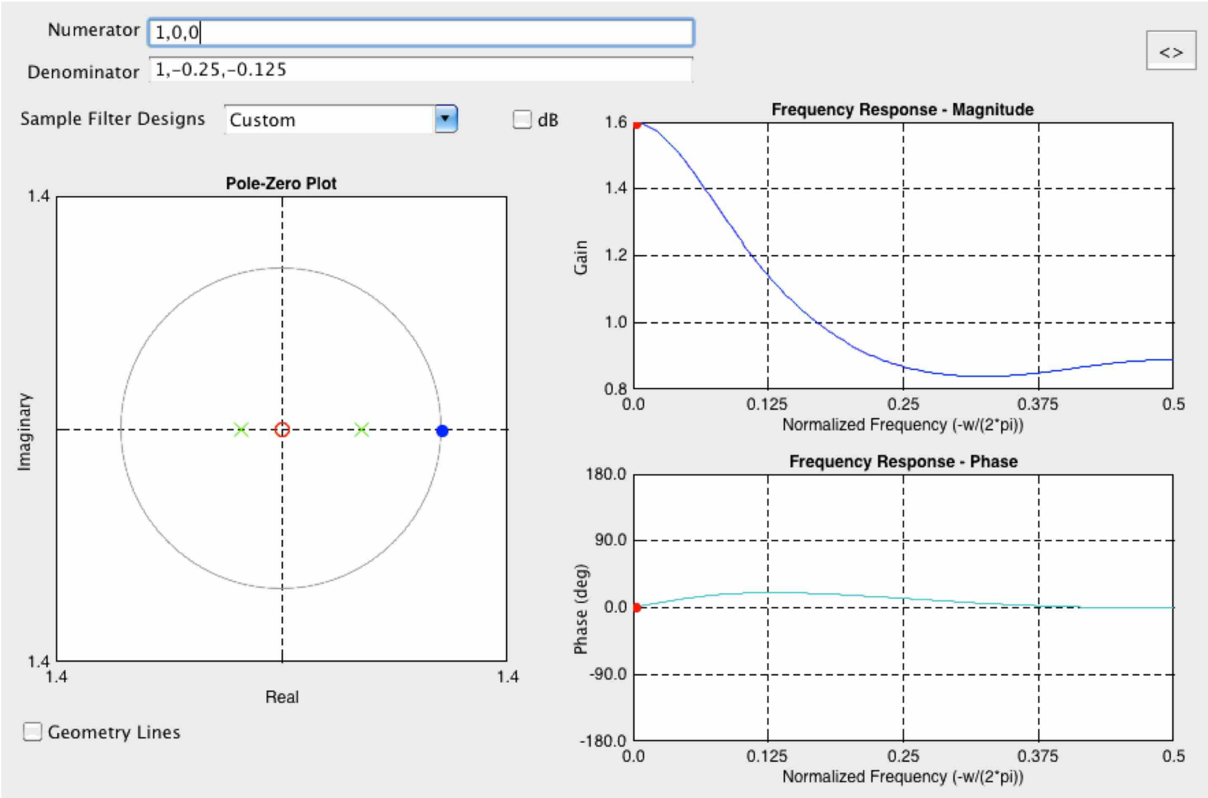


Example 3: H(z) = z^2 / [(z+1/4) (z-1/2)]



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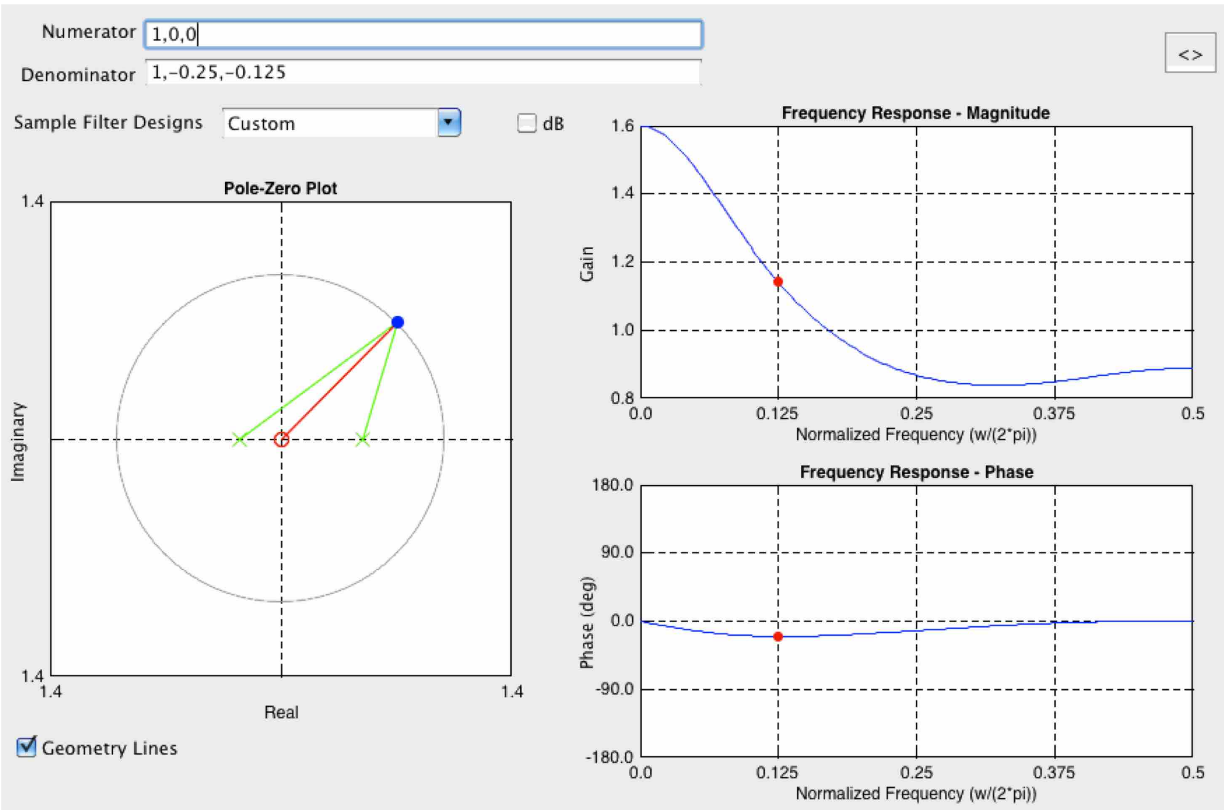

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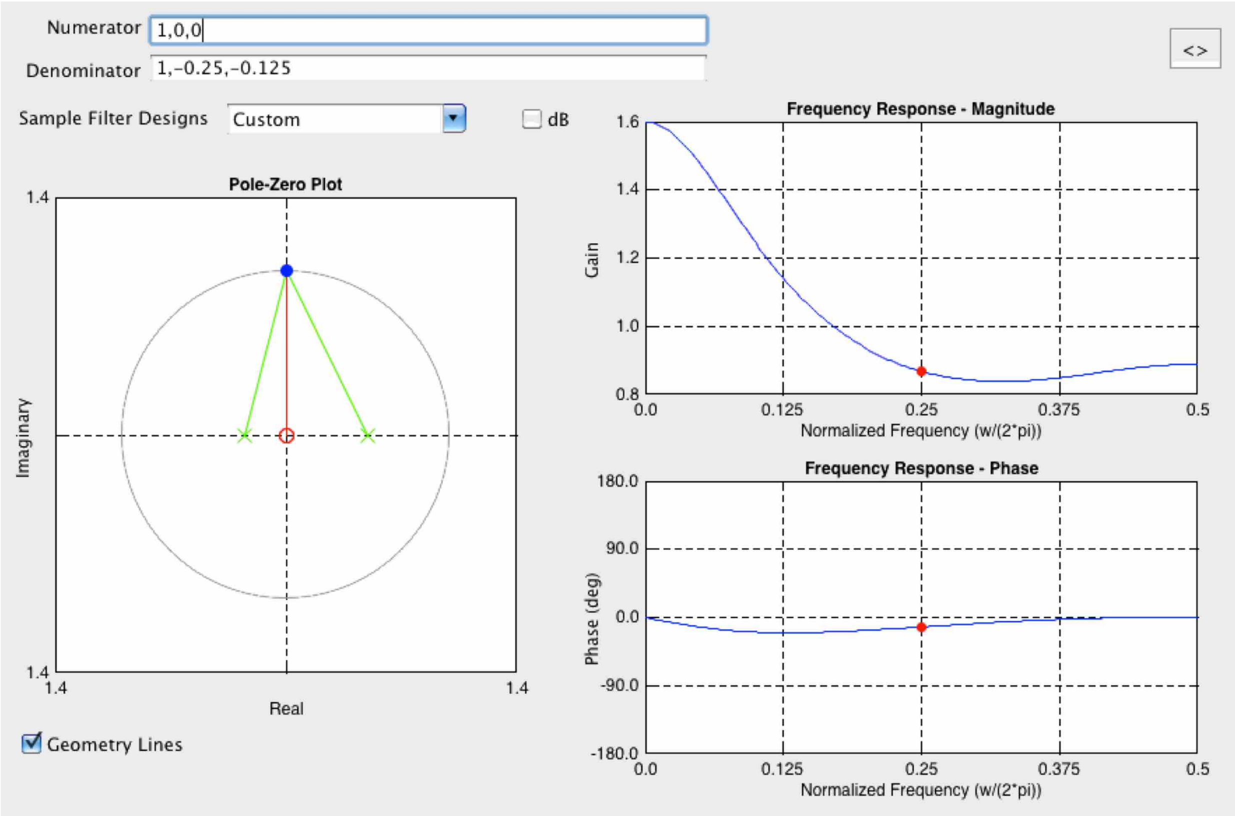
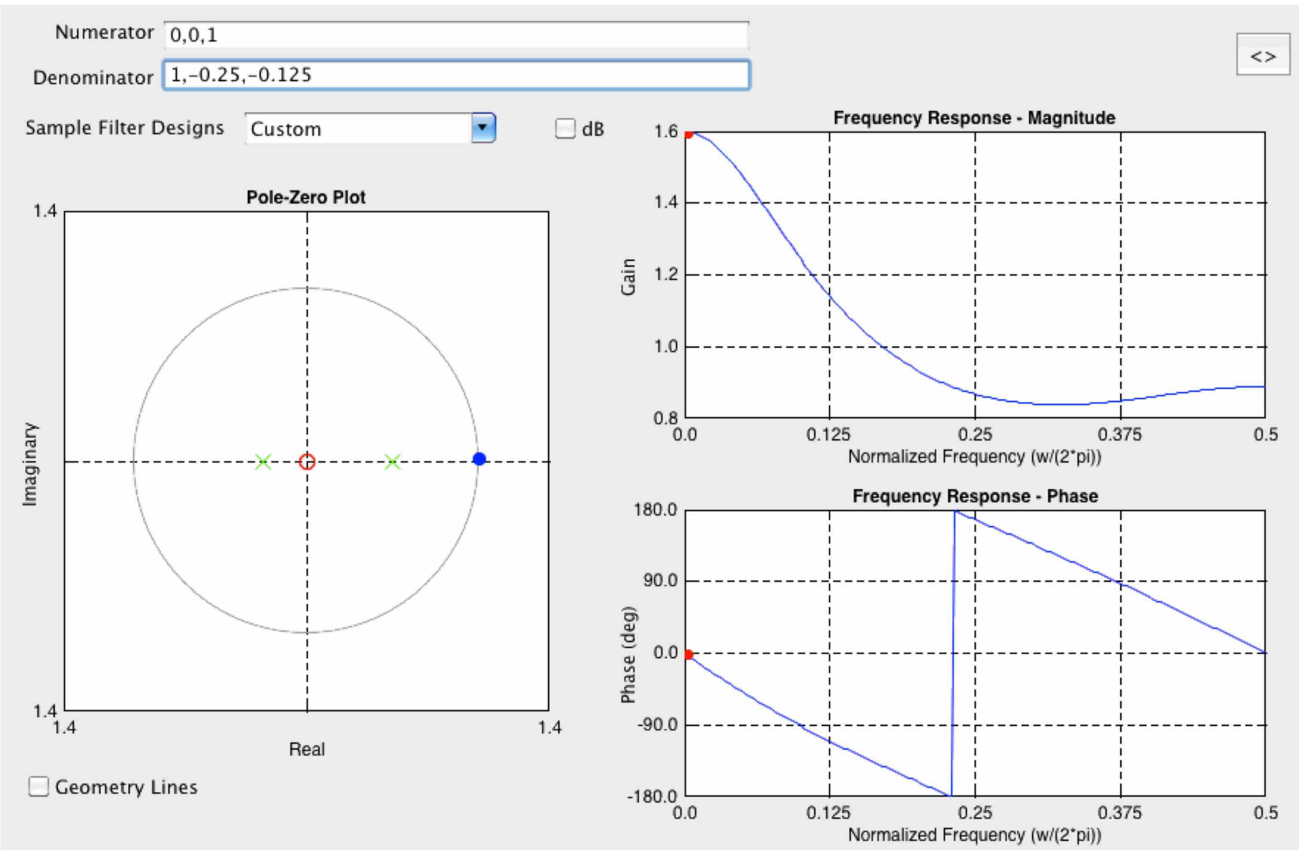


Example 4: H(z) = 1 / [(z+1/4) (z-1/2)]



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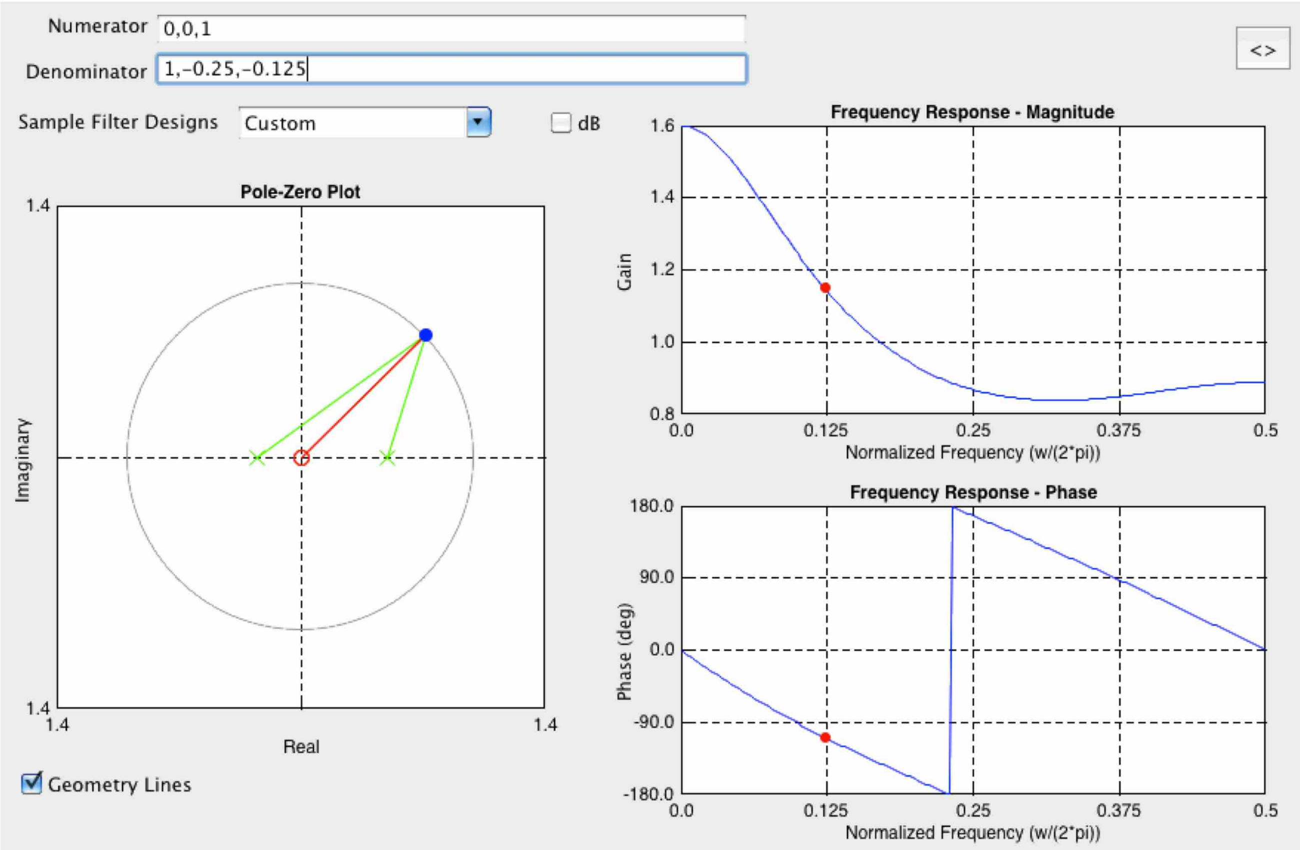

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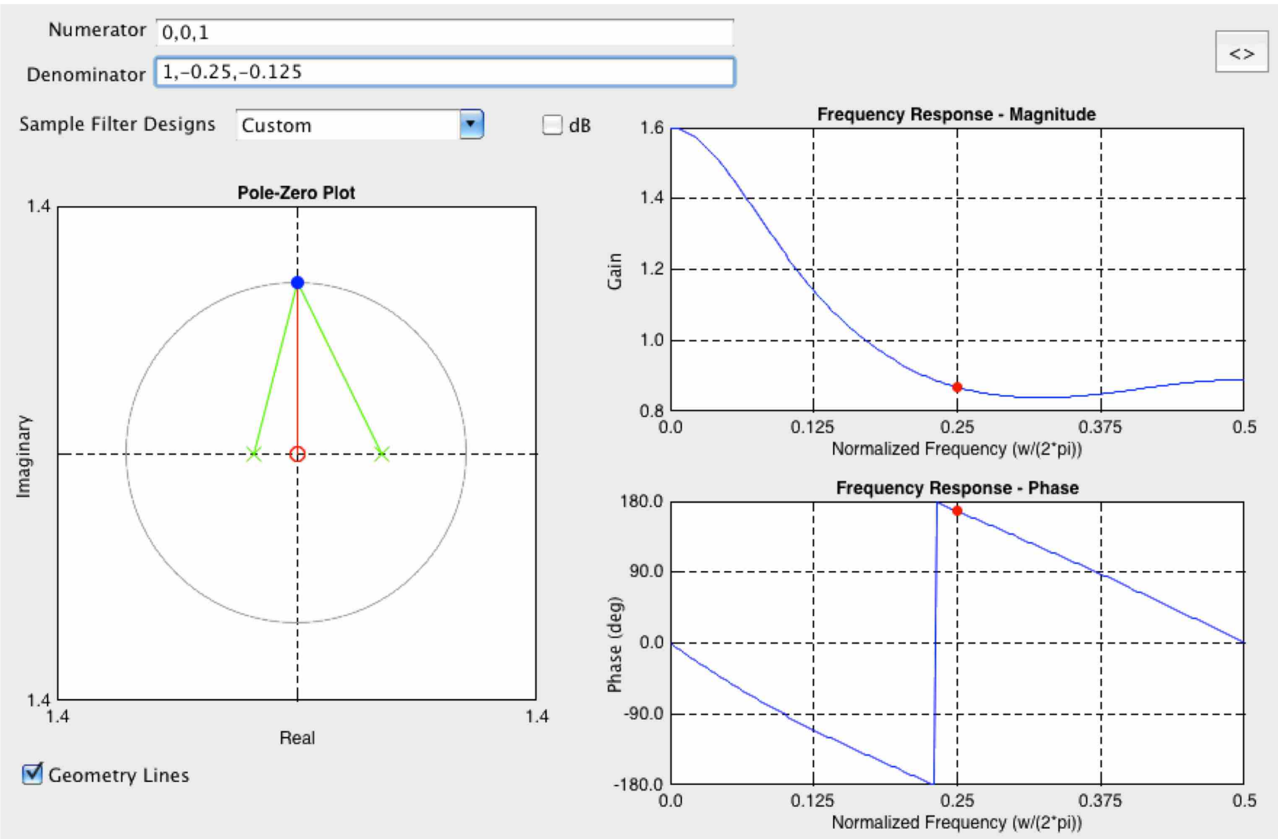


Example 5: H(z) = (z^2 - 1) / (z^3 +0.5z)

GEOMETRIC ALGORITHM FOR SKETCHING THE FREQUENCY RESPONSE

The Nth-order difference equation

$$y(n) - \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k x(n-k)$$

$$H(z) = \frac{z^{N-L}[b_0 z^L + b_1 z^{L-1} + \cdots + b_L]}{z^N - a_1 z^{N-1} - \cdots - a_N}$$

$$H(z) = b_0 z^{N-L} \frac{(z-n_1)(z-n_2)\cdots(z-n_L)}{(z-d_1)(z-d_2)\cdots(z-d_N)}$$

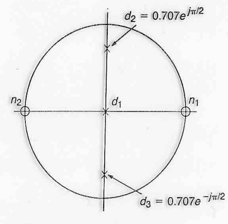
Keep in mind that the roots $n_1, n_2, \ldots, n_L$ (ZEROs) and $d_1, d_2, \ldots, d_N$ (POLEs) are complex numbers, in general, and that all **complex roots must appear in conjugate pairs** because the coefficients in the polynomials are all **real numbers.**

Example:

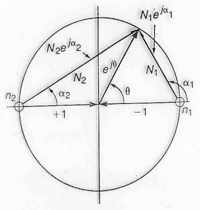
3rd order filter

$$H(z) = \frac{z^2 - 1}{z^3 + 0.5z}$$

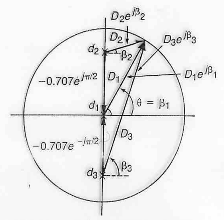
$$H(z) = \frac{(z-1)(z+1)}{z\,(z - 0.707\,e^{j\pi/2})(z - 0.707\,e^{-j\pi/2})}$$



(a) A plot of the poles and zeros



(b) A typical set of numerator vectors



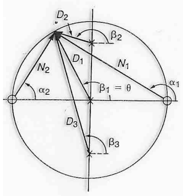
(c) A typical set of denominator vectors

Example 5: H(z) = (z^2 - 1) / (z^3 +0.5z)

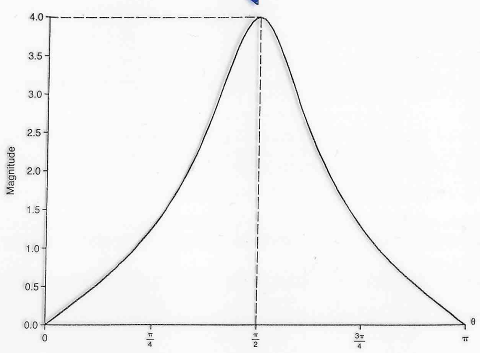
$$H(e^{j\theta}) = \frac{(e^{j\theta}-1)(e^{j\theta}+1)}{e^{j\theta}(e^{j\theta}-0.707e^{j\pi/2})(e^{j\theta}-0.707e^{-j\pi/2})}$$

$$H(e^{j\theta}) = \frac{(N_1e^{j\alpha_1})(N_2e^{j\alpha_2})}{(D_1e^{j\beta_1})(D_2e^{j\beta_2})(D_3e^{j\beta_3})}$$

$$H(e^{j\theta}) = \frac{N_1N_2}{(1)D_2D_3} e^{j(\alpha_1+\alpha_2-\theta-\beta_2-\beta_3)}$$



$$M_{\theta=\pi/2} = \frac{N_1N_2}{D_1D_2} \approx \frac{(\sqrt{2})(\sqrt{2})}{(0.3)(1.707)} \approx 4$$



PROCEDURE

We start with the factored form of the frequency response written in terms of the general complex variable z as

$$H(z) = \frac{b_0 z^{N-L}(z-n_1)(z-n_2)\cdots(z-n_L)}{(z-d_1)(z-d_2)\cdots(z-d_N)} \tag{5.26}$$

which in terms of $z = e^{j\theta}$ becomes

$$H(e^{j\theta}) = b_0 e^{j(N-L)\theta} \frac{(e^{j\theta}-n_1)(e^{j\theta}-n_2)\cdots(e^{j\theta}-n_L)}{(e^{j\theta}-d_1)(e^{j\theta}-d_2)\cdots(e^{j\theta}-d_N)}. \tag{5.27}$$

To interpret this graphically, Eq. 5.27 is written in terms of vectors as

$$H(e^{j\theta}) = b_0 e^{j(N-L)\theta} \frac{(N_1e^{j\alpha_1})(N_2e^{j\alpha_2})\cdots(N_Le^{j\alpha_L})}{(D_1e^{j\beta_1})(D_2e^{j\beta_2})\cdots(D_Ne^{j\beta_N})} \tag{5.28}$$

where $N_qe^{j\alpha_q}$ is the vector from the zero n_q to the tip of the $e^{j\theta}$ vector, $q = 1, 2, \ldots, L$ and $D_pe^{j\beta_p}$ is the vector from the pole d_p to the tip of the $e^{j\theta}$ vector, $p = 1, 2, \ldots, N$. Remember that $z = e^{j\theta}$ where θ is the digital frequency of the input. The magnitude of the frequency response is

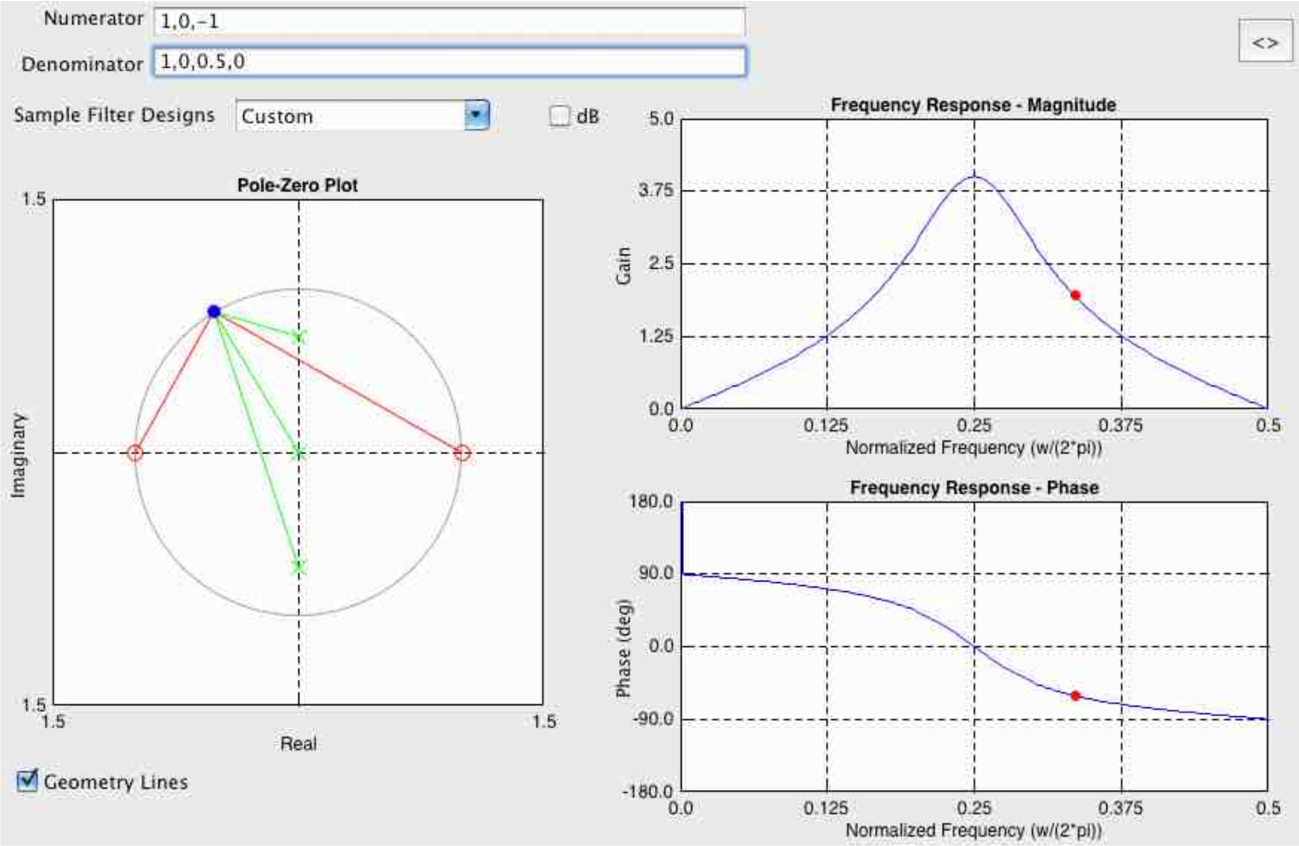
$$M = b_0N_1N_2\cdots N_L/D_1D_2\cdots D_N \tag{5.29}$$

and the phase is

$$P = (N-L)\theta + \alpha_1 + \alpha_2 + \cdots + \alpha_L - (\beta_1 + \beta_2 + \cdots + \beta_N) = (N-L)\theta + \sum_{k=1}^{L}\alpha_k - \sum_{k=1}^{N}\beta_k. \tag{5.30}$$

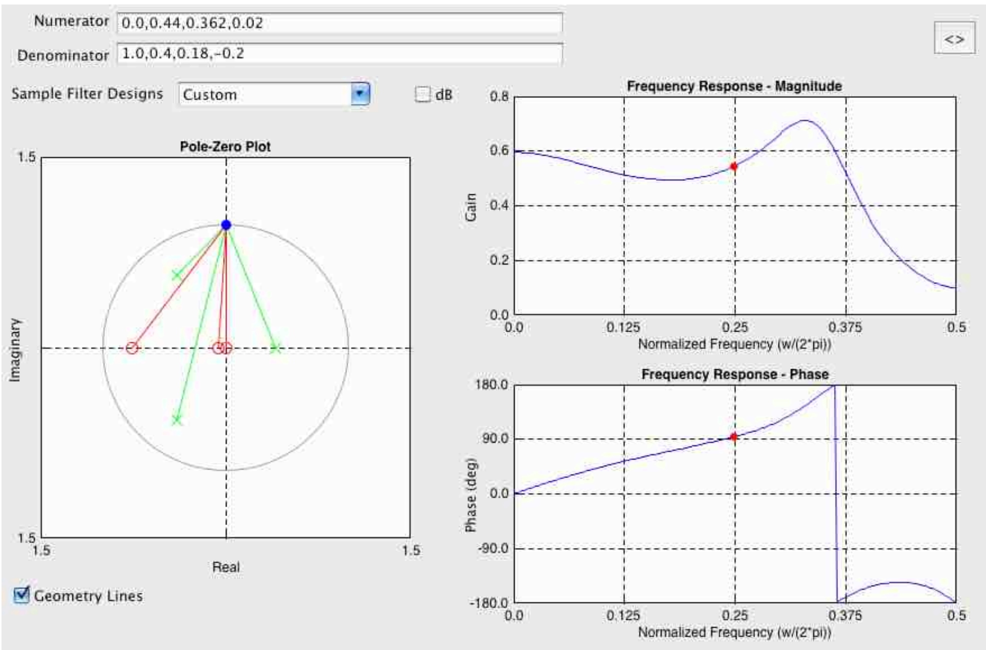
The N's, D's, α's, and β's can be evaluated as indicated in Fig. 5.13.

Example 5: H(z) = (z^2 - 1) / (z^3 +0.5z)



Example 6

$$H(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



FREQUENCY RESPONSE

Example 5.2. This example illustrates the graphical design of a bandpass filter.

A recursive second-order filter described by the difference equation (algorithm)

$$y(n) = a_1 y(n-1) + a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) \quad (5.54)$$

is to be used to implement a digital filter with the following specifications:

a) passband centered at about $\theta = \pi/2$.

b) unity gain at the center of the passband, i.e., $|H(e^{j\pi/2})| = 1$.

Design a filter to meet these requirements. That is, find the filter coefficients a_1, a_2, b_0, b_1, and b_2.

Solution: The recursive second-order filter of Eq. 5.54 has a generalized frequency response function of the form

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2} \quad (5.55)$$

which may be arranged in the factored form

$$H(z) = b_0 \frac{(z - n_1)(z - n_2)}{(z - d_1)(z - d_2)}. \quad (5.56)$$

The corresponding frequency response written as a ratio of vectors is

$$H(e^{j\theta}) = b_0 \frac{N_1 e^{j\alpha_1} N_2 e^{j\alpha_2}}{D_1 e^{j\beta_1} D_2 e^{j\beta_2}}. \quad (5.57)$$

Where should the poles and zeros be located? There are, of course, many solutions to a design problem that is posed as this one—with very general and somewhat unrestricted specifications. Let's proceed, however, with one possible solution.

i) The center of the bandpass is to be located close to $\theta = \pi/2$ radians. This means that the gain is to be "large" in the region of $\theta = \pi/2$ which can be achieved by making D_1 "small" near $\theta = \pi/2$. Thus, let us place a pole at the position d_1 shown in Fig. 5.16(a). Since there must also be a complex conjugate pole we also have one at d_2 as shown. We locate them more specifically in step iii.

ii) The poles d_1 and d_2 have been selected to control the bandpass characteristic of this filter and consequently the zeros n_1 and n_2 can be located in many suitable places, one of which is the origin where we put the two zeros as in Fig. 5.16(b). Now the magnitude of the frequency response is $M = b_0 N_1 N_2 / D_1 D_2 = b_0 / D_1 D_2$ because $N_1 = N_2 = 1$ for all values of θ.

iii) Lastly, to take care of the unity gain requirement at $\theta = \pi/2$ we draw two vectors (the two vectors from the zeros have already been accounted for) from d_1 and d_2 to $\theta = \pi/2$ on the unit circle as in Fig. 5.16(c). The gain at this value of $\theta = \pi/2$ is

$$M_{\pi/2} = 1 = b_0 \frac{1}{D_1 D_2} = b_0 \frac{1}{(1 - |d_1|)(1 + |d_2|)} = b_0 \frac{1}{1 - |d_1|^2} \quad \text{since } |d_1| = |d_2|. \quad (5.58)$$

We have one equation in two unknowns and if we choose $|d_1| = |d_2| = 0.9$ to get the poles close to the unit circle we have

$$1.0 = b_0/(1 - 0.9^2) \quad \text{or} \quad b_0 = 0.19 \quad (5.59)$$

and the unknown constants in Eq. 5.56 have been determined to be

$$n_1 = n_2 = 0, \; d_1 = 0.9e^{j\pi/2}, \; d_2 = 0.9e^{-j\pi/2}, \text{ and } b_0 = 0.19. \quad (5.60)$$

Now that we've selected the locations for the poles and zeros we want to find the filter coefficients so we can implement the filter. From Eqs. 5.56 and 5.60 the generalized frequency response is

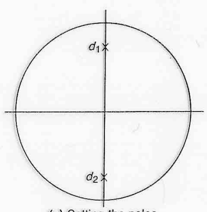
$$H(z) = \frac{0.19z^2}{(z - 0.9e^{j\pi/2})(z - 0.9e^{-j\pi/2})} = \frac{0.19z^2}{z^2 + 0.81} = \frac{0.19}{1 + 0.81z^{-2}} \quad (5.61)$$

and comparison with Eq. 5.55 yields the filter coefficients

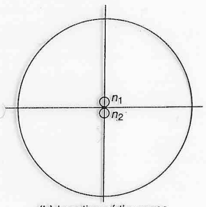
$$b_0 = 0.19, \; b_1 = b_2 = 0, \; a_1 = 0, \text{ and } a_2 = -0.81. \quad (5.62)$$

The filter realization and algorithm are given in Fig. 5.17(a) and a computer generated magnitude plot in Fig. 5.17(b). Notice the symmetry of the plot, due to the symmetrical poles and zeros, the DC gain of 0.105, and the desired gain of 1.0 at $\theta = \pi/2$.

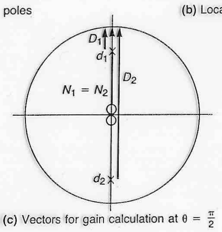
FREQUENCY RESPONSE



(a) Setting the poles

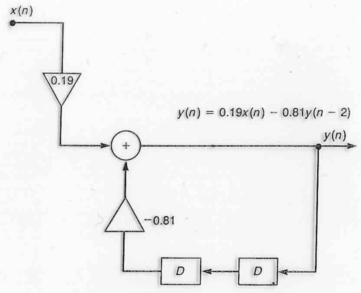


(b) Location of the zeros

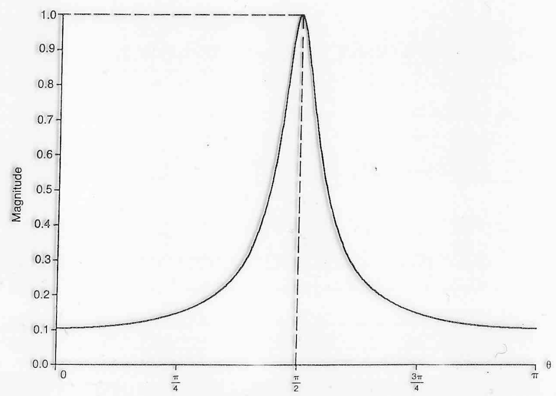


(c) Vectors for gain calculation at $\theta = \frac{\pi}{2}$

FIGURE 5.17 Results for Example 5.2



(a) An implementation of the bandpass design



(b) Frequency response magnitude, M

FREQUENCY RESPONSE

A second try

Comment: The shape of the plot is controlled by N_1N_2/D_1D_2 and the scale can be altered by adjusting b_0. Suppose we want a "sharper" characteristic than that in Fig. 5.17(b). We can get the magnitudes N_1 and N_2 into the act by placing two zeros at $n_1 = 0.9$ and $n_2 = -0.9$ as in Fig. 5.18(a). Now for the same gain at $\theta = \pi/2$ we have from Fig. 5.18(b)

$$M_{\pi/2} = 1.0 = b_0N_1N_2/D_1D_2 = b_0\frac{(1+0.81)^{1/2}(1+0.81)^{1/2}}{(0.1)(1.9)} \quad (5.63)$$

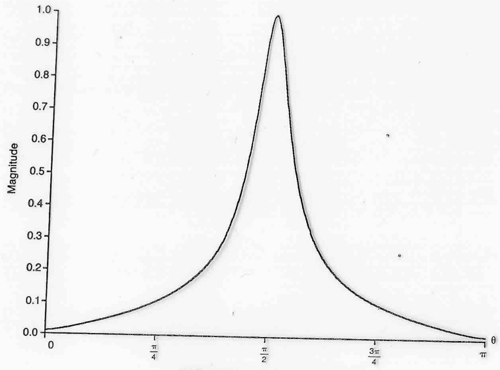
which yields $b_0 = 0.105$. The magnitude plot of Fig. 5.18(c) shows a much more selective bandpass characteristic and the generalized frequency response is

$$H(z) = 0.105\frac{(z+0.9)(z-0.9)}{z^2+0.81} = \frac{0.105z^2 - 0.085}{z^2+0.81} = \frac{0.105 - 0.085z^{-2}}{1+0.81z^{-2}}. \quad (5.64)$$

Comparison with Eq. 5.55 yields the filter coefficients

$$b_0 = 0.105,\ b_1 = 0,\ b_2 = -0.085,\ a_1 = 0,\ \text{and}\ a_2 = -0.81 \quad (5.65)$$

with the filter implementation in Fig. 5.18(d).



(c) Frequency response magnitude, M

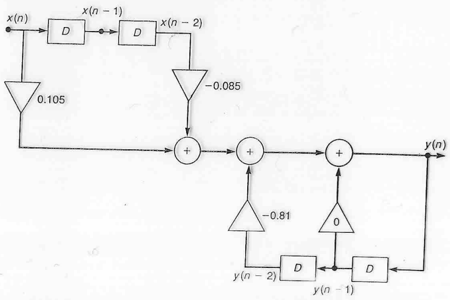
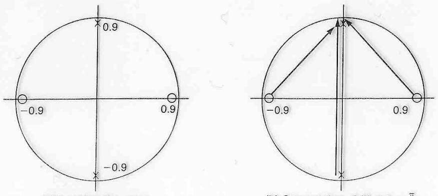


FIGURE 5.18 Design of a bandpass filter (a second try)



(a) Locating the zeros

(b) Computation of M at $\theta = \frac{\pi}{2}$

(d) An implementation of $y(n) = -0.81y(n-2) + 0.105x(n) - 0.085x(n-2)$

FREQUENCY RESPONSE

$\omega = \Omega \cdot T = 2\pi 60/360 = 2\pi/6 = \pi/3$

Example 5.4. This example illustrates the graphical design of a comb filter.

In medical applications, the 60-Hz frequency of the power supply is often "picked up" by test equipment such as an EKG recorder. At the same time, harmonically related frequencies such as $f_2 = 2 \times 60 = 120$ Hz and $f_3 = 3 \times 60 = 180$ Hz are created because of nonlinear phenomena. To ensure an accurate recording, it is necessary to eliminate or suppress these frequencies as the unfinished magnitude plot of Fig. 5.20(a) suggests. Design a nonrecursive digital filter that will accomplish this, given a sampling frequency of 360 Hz. → $F_s = 360$ Hz

Solution: The pertinent digital frequencies are calculated from $\omega = 2\pi f/f_s$ and they are

$$\omega_1 = 2\pi(60)/360 = \pi/3,\ \omega_2 = 2\pi(120)/360 = 2\pi/3,\ \text{and}\ \omega_3 = 2\pi(180)/360 = \pi. \quad (5.74)$$

We know that a zero placed on the unit circle at $\omega = \omega_k$ eliminates the digital frequency ω_k, which accounts for the three zeros at $\omega = \pi/3, 2\pi/3$, and π in Fig. 5.20(b). Remember that complex zeros must occur in conjugate pairs, and this requires the addition of the two zeros at $\omega = -\pi/3$ and $-2\pi/3$. The sixth zero at $\omega = 0$ is added to eliminate any DC in the signal and also creates a symmetrical pattern which produces the often desirable property of linear phase. ! →

Our ultimate goal is the algorithm to implement this filter so we first need to form $H(z) = N(z)/D(z)$ and create the difference equation from this frequency response function. From Fig. 5.20(b) the numerator polynomial is

$$N(z) = (z-1)(z-e^{j\pi/3})(z-e^{j2\pi/3})(z-e^{j\pi})(z-e^{j4\pi/3})(z-e^{j5\pi/3}) \quad (5.75)$$

which when multiplied and simplified yields the expression

$$N(z) = z^6 - 1. \quad (5.76)$$

What about the denominator $D(z)$? For simplicity, we might be tempted to make $D(z) = 1$ which would give

$$H(z) = z^6 - 1 \quad (5.77)$$

and the corresponding difference equation

$$y(n) = x(n+6) - x(n) \quad (5.78)$$

which describes a noncausal filter because of the requirement for $x(n+6)$, the sixth future value of the input which is, of course, unknown to all but certain stargazers and several pseudosavants. Instead, by making $D(z) = z^6$ we have

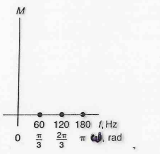
$$H(z) = \frac{z^6-1}{z^6} = 1 - z^{-6} \quad (5.79)$$

and the difference equation for a causal filter

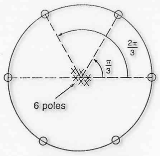
$$y(n) = x(n) - x(n-6) \quad (5.80)$$

giving the implementation of Fig. 5.20(c). Because of the symmetrical placing of the zeros, and because all of the poles are at the origin $(z = 0)$, the symmetrical response of Fig. 5.20(d) results.

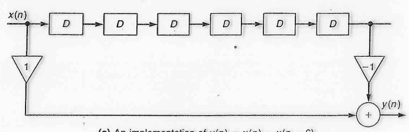
FIGURE 5.20 Nonrecursive (comb) filter design



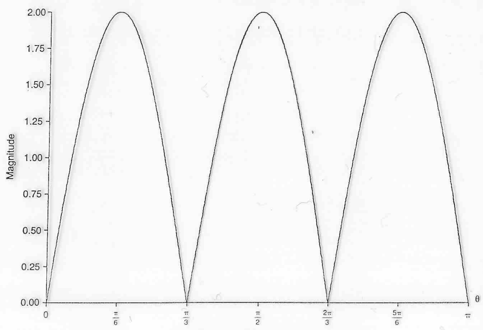
(a) Desired gains of zero at f = 60, 120 and 180 Hertz



(b) Locating six zeros



(c) An implementation of $y(n) = x(n) - x(n-6)$



(d) Magnitude of frequency response

$H(z) = 1 - z^{-6}$

$\frac{Y(z)}{X(z)} = 1 - z^{-6} \Rightarrow Y(z) = X(z) - z^{-6}X(z) \to y(n) = x(n) - x(n-6)$

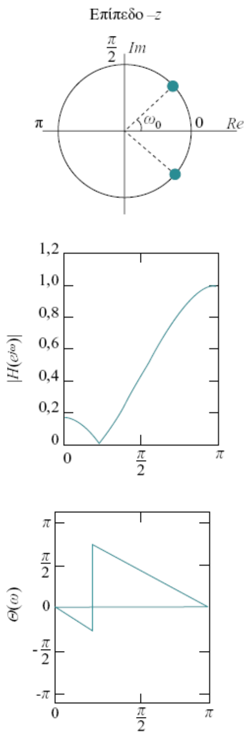
Φίλτρα Εγκοπής (Notch Filters)

Παράδειγμα 3.10

Να σχεδιαστεί ψηφ. φίλτρο που να αποκόπτει τη συχνότητα ω0

$$z_{1,2} = 1e^{\pm j\omega_0}$$

$$H(e^{j\omega}) = b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1}) = b_0(1 - 2\cos\omega_0 z^{-1} + z^{-2})$$



(α)

Φίλτρα Εγκοπής (Notch Filters)

Παράδειγμα 3.10 (συνέχεια…)

$$z_{1,2} = 1e^{\pm j\omega_0}$$

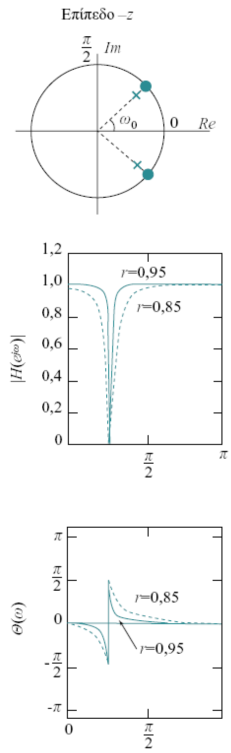
$$H(e^{j\omega}) = b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1}) = b_0(1 - 2\cos\omega_0 z^{-1} + z^{-2})$$

Τοποθετούμε ζεύγος συζυγών μιγαδικών πόλων:

$$p_{1,2} = re^{\pm j\omega_0} \qquad 0 < r < 1$$

$$H(e^{j\omega}) = b_0 \frac{(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

$$= b_0 \frac{1 - 2\cos\omega_0 z^{-1} + z^{-2}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

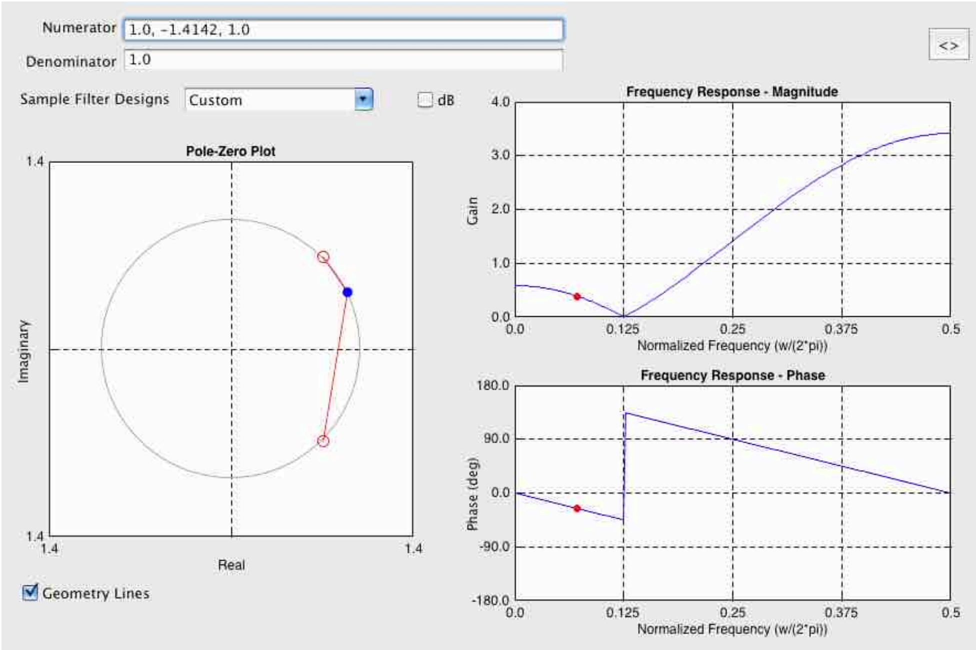


(β)

Φίλτρα Εγκοπής (Notch Filters)

Παράδειγμα 3.10 (συνέχεια…)

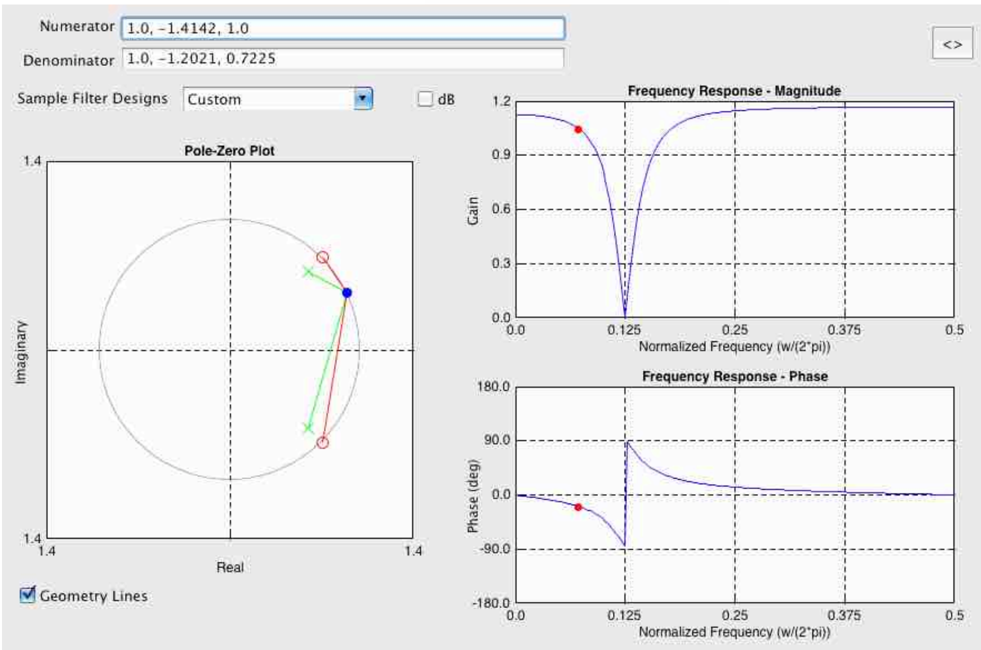
Φίλτρο εγκοπής (notch) με ένα μόνο μηδενικό (και το συζυγές του) στη συχνότητα ω=π/4



Φίλτρα Εγκοπής (Notch Filters)

Παράδειγμα 3.10 (συνέχεια…)

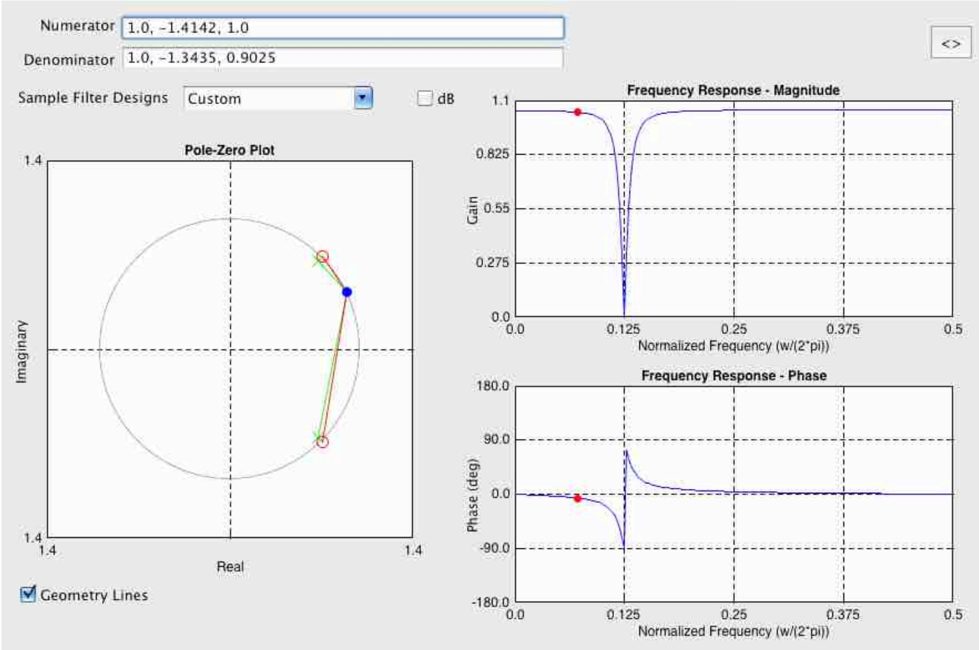
Φίλτρο εγκοπής (notch) με μηδενικό και πόλο σε απόσταση **r=0.85** (και τα συζυγή τους) στη συχνότητα ω=π/4



Φίλτρα Εγκοπής (Notch Filters)

Παράδειγμα 3.10 (συνέχεια…)

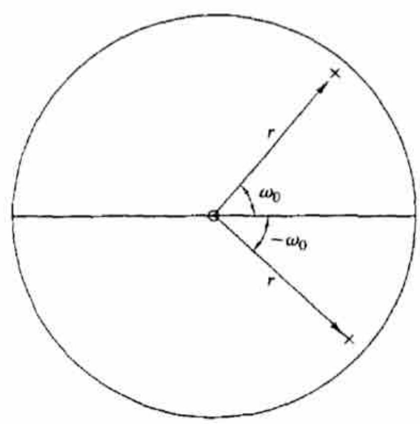
Φίλτρο εγκοπής (notch) με μηδενικό και πόλο σε απόσταση **r=0.95** (και τα συζυγή τους) στη συχνότητα ω=π/4



Digital Resonators

A special **two-pole** bandpass filter with the pair of complex-conjugate poles located near the unit circle.

Useful in simple **bandpass filtering** and **speech generation.**



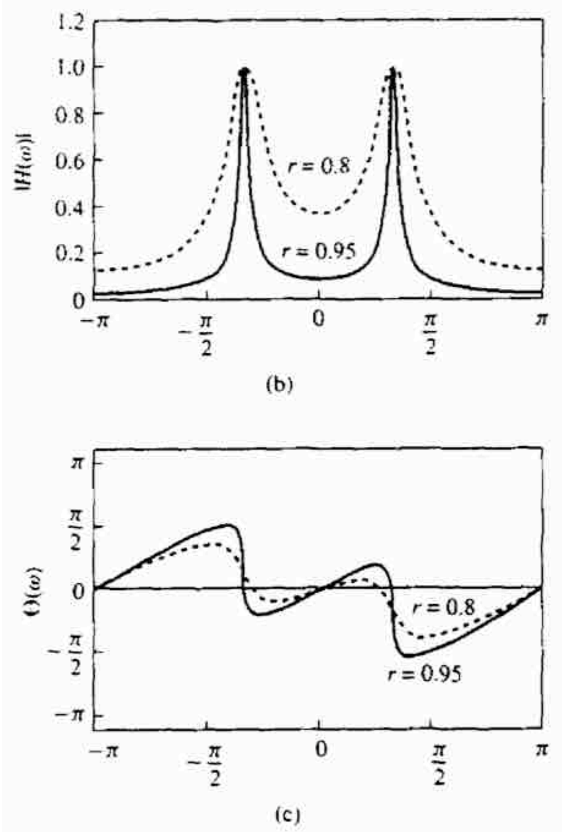
$$p_{1,2} = re^{\pm j\omega_0} \qquad 0 < r < 1$$

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

$$H(z) = \frac{b_0}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

Digital Resonators (cont…)

Zeros at z = 0

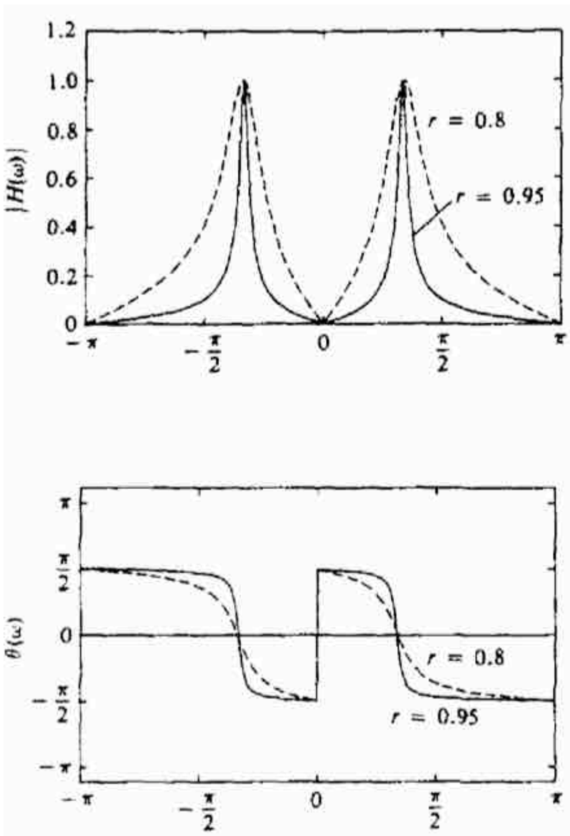


Digital Resonators (cont…)

Zeros at z = 1, z = -1

$$H(z) = G\frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

$$= G\frac{1-z^{-2}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$$



Digital Sinusoidal Oscillators

Resonators whose **poles** are located **on the unit circle**.

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

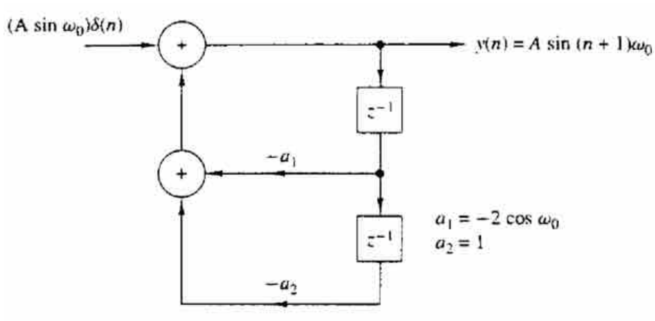
$$a_1 = -2r\cos\omega_0 \qquad a_2 = r^2$$

$$p = re^{\pm j\omega_0}$$

$$h(n) = \frac{b_0 r^n}{\sin\omega_0}\sin(n+1)\omega_0 u(n)$$

$$r = 1$$

$$h(n) = A\sin(n+1)\omega_0 u(n)$$



Digital Sinusoidal Oscillators (cont…)