



ΕΠΕΞΕΡΓΑΣΙΑ ΣΗΜΑΤΩΝ

ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΙ

ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΣ FOURIER 2D & 3D ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΣ HARTLEY

ΚΑΘΗΓΗΤΗΣ ΑΘΑΝΑΣΙΟΣ ΣΚΟΔΡΑΣ

skodras@upatras.gr
www.ece.upatras.gr/skodras

Σχέσεις Μετασχηματισμών Fourier

	ΣΥΝΕΧΟΥΣ ΧΡΟΝΟΥ	ΔΙΑΚΡΙΤΟΥ ΧΡΟΝΟΥ
ΠΕΡΙΟΔΙΚΟ	<p>Σειρά Fourier</p> $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\Omega_0 t}$ $\alpha_k = \frac{1}{T} \int_T x(t) e^{-jk\Omega_0 t} dt$	<p>Σειρά Fourier</p> $x(n) = \sum_{k=0}^{N-1} X(k) W_N^{-nk}$ $X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) W_N^{nk}$ $W_N = e^{-j\frac{2\pi}{N}}$
ΜΗ-ΠΕΡΙΟΔΙΚΟ	<p>Μετασχηματισμός Fourier</p> $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$ $X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	<p>Μετασχηματισμός Fourier</p> $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

ΣΥΝΕΧΗΣ ΧΡΟΝΟΣ / ΧΩΡΟΣ

Μετασχηματισμός Fourier 1-Διάστασης

Ευθείς:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Αντίστροφος:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

Μετασχηματισμός Fourier 2-Διαστάσεων

$$f(x, y) \xleftrightarrow{2D-F} F(\omega_x, \omega_y)$$

Ευθύς:
$$F(\omega_x, \omega_y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{-j(\omega_x x + \omega_y y)} dx dy$$

Αντίστροφος:
$$f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega_x, \omega_y) e^{j(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

Ιδιότητες

Γραμμικότητα: Αν $f(x, y) \xleftrightarrow{2D-F} F(\omega_x, \omega_y)$, $g(x, y) \xleftrightarrow{2D-F} G(\omega_x, \omega_y)$

τότε $a f(x, y) + b g(x, y) \xleftrightarrow{2D-F} a F(\omega_x, \omega_y) + b G(\omega_x, \omega_y)$

Κλιμάκωση: Αν $f(x, y) \xleftrightarrow{2D-F} F(\omega_x, \omega_y)$

τότε $f(ax, by) \xleftrightarrow{2D-F} \frac{1}{|ab|} F\left(\frac{\omega_x}{a}, \frac{\omega_y}{b}\right)$

Ολιγόθρον
στον χώρο:

$$f(x-x_0, y-y_0) \xleftrightarrow{2D-F} e^{-j(\Omega_x x_0 + \Omega_y y_0)} F(\Omega_x, \Omega_y)$$

Ολιγόθρον
στη συχνότητα:

$$e^{j(\Omega_x x + \Omega_y y)} f(x, y) \xleftrightarrow{2D-F} F(\Omega_x - \Omega_1, \Omega_y - \Omega_2)$$

Συνέλιξη:

$$f(x, y) ** g(x, y) \xleftrightarrow{2D-F} F(\Omega_x, \Omega_y) \cdot G(\Omega_x, \Omega_y)$$

Γινόμενο:

$$f(x, y) \cdot g(x, y) \xleftrightarrow{2D-F} \frac{1}{(2\pi)^2} F(\Omega_x, \Omega_y) ** G(\Omega_x, \Omega_y)$$

Ανεξαρτησία: Αν $f_1(x) \xrightarrow{F} F_1(\Omega_x)$, $f_2(y) \xrightarrow{F} F_2(\Omega_y)$

τότε $f_1(x) f_2(y) \xleftrightarrow{2D-F} F_1(\Omega_x) \cdot F_2(\Omega_y)$

Τύπος Rayleigh: $\iint_{-\infty}^{\infty} |f(x, y)|^2 dx dy = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} |F(\Omega_x, \Omega_y)|^2 d\Omega_x d\Omega_y$

Θεώρημα περιστροφής: Αν η συνάρτηση $f(x, y)$ περιστραφεί στο επίπεδο (x, y) κατά γωνία θ , τότε και ο 2D μετασχηματισμός Fourier περιστρέφεται στο επίπεδο (Ω_x, Ω_y) κατά την ίδια γωνία θ .

$$f(x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) \xleftrightarrow{2D-F} F(\Omega_x \cos \theta - \Omega_y \sin \theta, \Omega_x \sin \theta + \Omega_y \cos \theta)$$

Μετασχηματισμός Fourier 3-Διαστάσεων

$$f(x, y, z) \xleftrightarrow{3D-F} F(\omega_x, \omega_y, \omega_z)$$

$$F(\omega_x, \omega_y, \omega_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-j(\omega_x x + \omega_y y + \omega_z z)} dx dy dz$$

$$f(x, y, z) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_x, \omega_y, \omega_z) e^{j(\omega_x x + \omega_y y + \omega_z z)} d\omega_x d\omega_y d\omega_z$$

ΔΙΑΚΡΙΤΟΣ ΧΡΟΝΟΣ / ΧΩΡΟΣ

DTFT 2-ΔΙΑΣΤΑΣΕΩΝ

Έστω το 2-διάστατων διακριτών χρόνου σήμα $x(m, n)$, όπου m, n διακριτές μεταβλητές. Ο μετασχηματισμός Fourier διακριτών χρόνου για το σήμα αυτό, ορίζεται ως

$$X(e^{j\omega_m}, e^{j\omega_n}) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m, n) e^{-j(\omega_m m + \omega_n n)} \quad (1)$$

όπου ω_m, ω_n οι κυλιόμενες συχνότητες κατά μήκος των m, n αντίστοιχα.

Ο αντίστροφος DTFT 2-διάστατων ισούται με

$$x(m, n) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\omega_m}, e^{j\omega_n}) e^{j\omega_m m} e^{j\omega_n n} d\omega_m d\omega_n$$

Ο υπολογισμός της σχέσης (1) μπορεί να γίνει ως δύο διαδοχικοί μονοδιάστατοι μετασχηματισμοί Fourier, πρώτα κατά m και μετά κατά n .

Απόδειξη:

$$\begin{aligned} (1) \rightarrow X(e^{j\omega_m}, e^{j\omega_n}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m, n) e^{-j(\omega_m m + \omega_n n)} \\ &= \sum_{n=-\infty}^{\infty} \left[\underbrace{\sum_{m=-\infty}^{\infty} x(m, n) e^{-j\omega_m m}}_{X(e^{j\omega_m}, n)} \right] e^{-j\omega_n n} \\ &= \sum_{n=-\infty}^{\infty} X(e^{j\omega_m}, n) e^{-j\omega_n n} \quad (2) \end{aligned}$$

$$\text{όπου } X(e^{j\omega_m}, n) = \sum_{m=-\infty}^{\infty} x(m, n) e^{-j\omega_m m} \quad (3)$$

Αντίστροφος

$$x(m,n) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{j\omega_m}, e^{j\omega_n}) e^{j\omega_m m} e^{j\omega_n n} d\omega_m d\omega_n$$

Απόδειξη

Από τη σχέση (3) μπορούμε να υπολογίσουμε το σήμα $x(m,n)$ με βάση τον μονοδιάστατο αντίστροφο μετασχηματισμό του $X(e^{j\omega_m}, n)$:

$$x(m,n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega_m}, n) e^{j\omega_m m} d\omega_m \quad (5)$$

Αλλά από τη σχέση (2) το $X(e^{j\omega_m}, n)$ μπορεί να προκύψει ως αντίστροφος μονοδιάστατος μετασχηματισμός του $X(e^{j\omega_m}, e^{j\omega_n})$:

$$X(e^{j\omega_m}, n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega_m}, e^{j\omega_n}) e^{j\omega_n n} d\omega_n \quad (6)$$

Αντικαθιστώντας την (6) στην (5) προκύπτει η (4).

ΑΣΚΗΣΗ Έστω $x(m,n) = x_1(m) x_2(n)$. Να υπολογιστεί ο DTFT του σήματος $x(m,n)$, όταν γνωρίζουμε ότι ο DTFT του μονοδιάστατου σήματος $x_1(m)$ είναι $X_1(e^{j\omega_m})$ και εκείνος του $x_2(n)$ είναι $X_2(e^{j\omega_n})$.

ΛΥΣΗ

$$\begin{aligned} X(e^{j\omega_m}, e^{j\omega_n}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m,n) e^{-j(\omega_m m + \omega_n n)} = \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m) x_2(n) e^{-j\omega_m m} e^{-j\omega_n n} = \\ &= \sum_{m=-\infty}^{\infty} x_1(m) e^{-j\omega_m m} \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\omega_n n} = \\ &= X_1(e^{j\omega_m}) \cdot X_2(e^{j\omega_n}) \end{aligned}$$

Εφαρμογή 1 Έστω $x(m,n) = \delta(m-1)\delta(n+4)$. Με βάση το παραπάνω κριτήριο

θα έχουμε ότι:

$$X(e^{j\omega_m}, e^{j\omega_n}) = e^{-j\omega_m} \cdot e^{j4\omega_n}$$

αφού $\delta(m-1) \xleftrightarrow{F} e^{-j\omega_m}$ και $\delta(n+4) \xleftrightarrow{F} e^{j\omega_n 4}$

② Έστω $x(m,n) = \left(\frac{1}{2}\right)^{n-m} u(n-2)u(-m)$. Η σχέση αυτή γράφεται ως

$$x(m,n) = \underbrace{\left(\frac{1}{2}\right)^{n-2} u(n-2)}_{x_2(n)} \left(\frac{1}{2}\right)^2 \underbrace{\left(\frac{1}{2}\right)^{-m} u(-m)}_{x_1(m)}$$

$$\begin{array}{ccc} \uparrow F & & \swarrow F \\ e^{-j\omega_n 2} & \frac{1}{1 - \frac{1}{2}e^{-j\omega_n}} & \left(\frac{1}{2}\right)^2 \frac{1}{1 - \frac{1}{2}e^{+j\omega_m}} \end{array}$$

$$\text{Άρα } X(e^{j\omega_m}, e^{j\omega_n}) = \left[\frac{e^{-j2\omega_n}}{1 - \frac{1}{2}e^{-j\omega_n}} \right] \left[\frac{1}{4} \frac{1}{1 - \frac{1}{2}e^{+j\omega_m}} \right]$$

ΑΣΚΗΣΗ Να υπολογιστεί ο DTFT του σήματος $x(m,n) = \left(\frac{1}{2}\right)^n \cos\left(\frac{2\pi n}{3}\right) u(n)$.

ΛΥΣΗ Το διδιάστατο σήμα $x(m,n)$ μπορεί να εκφραστεί ως γινόμενο δύο μονοδιάστατων σήματων:

$$x(m,n) = \underbrace{\left(\frac{1}{2}\right)^n u(n)}_{x_2(n)} \cdot \underbrace{\cos\left(\frac{2\pi}{3}n\right)}_{x_1(m)} \quad (1)$$

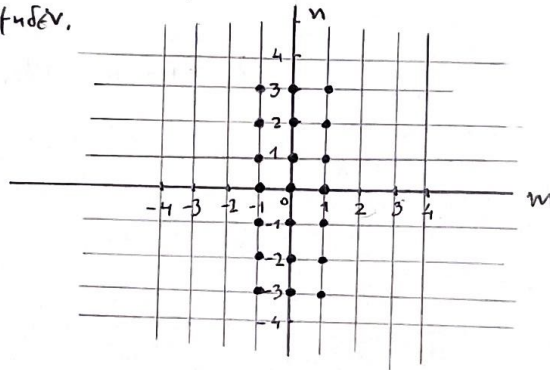
$$F\{x_2(n)\} = F\left\{\left(\frac{1}{2}\right)^n u(n)\right\} = \frac{1}{1 - \frac{1}{2}e^{-j\omega_n}} \quad (2)$$

$$F\{x_1(m)\} = F\left\{\cos\left(\frac{2\pi}{3}n\right)\right\} = \pi \left[\sum_{k=-\infty}^{\infty} \delta\left(\omega_m - \frac{2\pi}{3} - 2k\pi\right) + \sum_{k=-\infty}^{\infty} \delta\left(\omega_m + \frac{2\pi}{3} - 2k\pi\right) \right] \quad (3)$$

Άρα ο DTFT $X(e^{j\omega_m}, e^{j\omega_n})$ του σήματος $x(m,n)$ ισούται με το γινόμενο των (2) και (3).

ΑΣΚΗΣΗ Να υπολογιστεί ο DTFT του σήματος $x(m,n) = \begin{cases} 1, & -2 \leq m \leq 2 \wedge -4 \leq n \leq 4 \\ 0, & \text{άλλοι} \end{cases}$

ΛΥΣΗ Το σήμα $x(m,n)$ έχει τιμή 1 στα έντονα σφαιράκια του πλέγματος. Στα υπόλοιπα σφαιράκια είναι μηδέν.



Το σήμα $x(m,n)$ μπορεί να εκφραστεί ως γινόμενο διακριτών κεραιωτικών.

$$x(m,n) = \underbrace{[u(m+1) - u(m-2)]}_{x_1(m)} \underbrace{[u(n+3) - u(n-4)]}_{x_2(n)} \quad (1)$$

$$\begin{array}{ccc} \uparrow F & & \uparrow F \\ \frac{\sin(\frac{3\omega m}{2})}{\sin(\frac{\omega m}{2})} & & \frac{\sin(\frac{7\omega n}{2})}{\sin(\frac{\omega n}{2})} \end{array}$$

Τελικά

$$X(e^{j\omega m}, e^{j\omega n}) = \left[\frac{\sin(\frac{3\omega m}{2})}{\sin(\frac{\omega m}{2})} \right] \left[\frac{\sin(\frac{7\omega n}{2})}{\sin(\frac{\omega n}{2})} \right]$$

Σημείωση: Ουθυθετείτε ότι ο DTFT του σήματος $g(n) = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{άλλοι} \end{cases}$

$$\text{ισχύει } G(e^{j\omega}) = \frac{\sin(\frac{2M+1}{2}\omega)}{\sin(\frac{\omega}{2})}$$

ΑΣΚΗΣΗ Να υπολογιστεί ο DTFT του σήματος $x(m, n) = \sin\left(\frac{2\pi}{5}m + \frac{\pi}{3}n\right)$

ΛΥΣΗ Για το φασοειδές σήμα $g(n) = e^{j\omega_0 n}$ γνωρίζουμε ότι ο DTFT ισοδύναμο είναι

$$G(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi\ell) \quad (1)$$

Το δεδομένο σήμα $x(m, n)$ ανάλυται ως εξής:

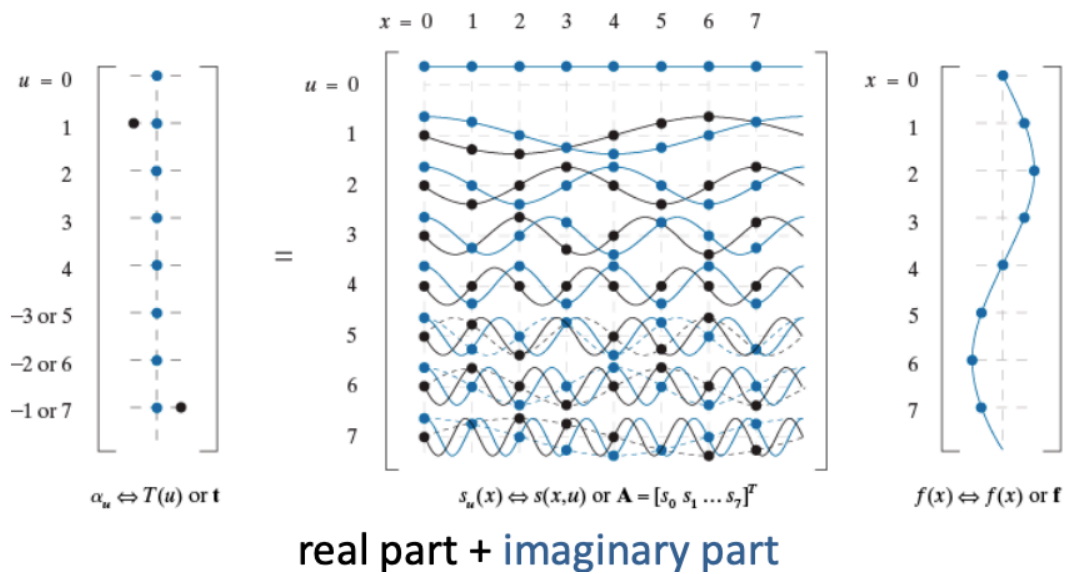
$$\begin{aligned} x(m, n) &= \sin\left(\frac{2\pi}{5}m + \frac{\pi}{3}n\right) = \frac{1}{2j} \left[e^{j\left(\frac{2\pi}{5}m + \frac{\pi}{3}n\right)} - e^{-j\left(\frac{2\pi}{5}m + \frac{\pi}{3}n\right)} \right] = \\ &= \frac{1}{2j} \left[e^{j\frac{2\pi}{5}m} e^{j\frac{\pi}{3}n} - e^{-j\frac{2\pi}{5}m} e^{-j\frac{\pi}{3}n} \right] \\ &\quad \begin{array}{l} \uparrow \text{F} \\ 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega_m - \frac{2\pi}{5} - 2\pi k\right) \\ \uparrow \text{F} \\ 2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega_n - \frac{\pi}{3} - 2\pi\ell\right) \end{array} \quad \begin{array}{l} \uparrow \text{F} \\ 2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\omega_n + \frac{\pi}{3} - 2\pi\ell\right) \\ \uparrow \text{F} \\ 2\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega_m + \frac{2\pi}{5} - 2\pi k\right) \end{array} \end{aligned}$$

Τελικά ο DTFT του σήματος $x(m, n)$ ισοδύναμο είναι:

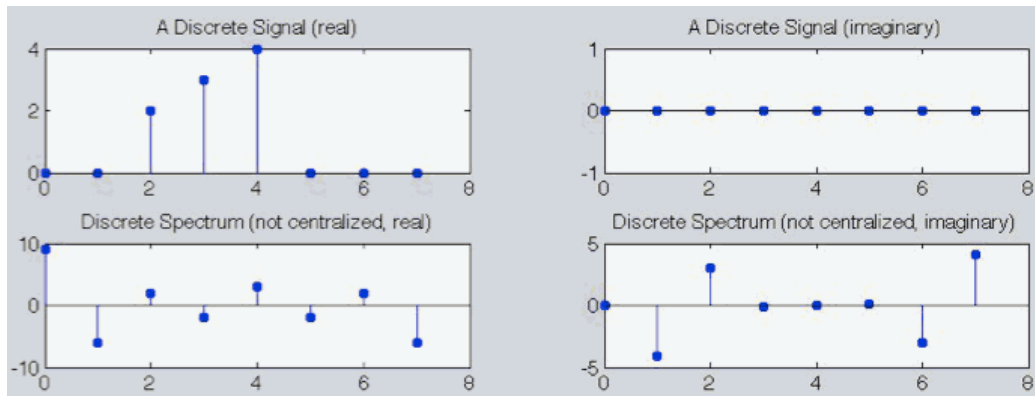
$$\begin{aligned} X(e^{j\omega_m}, e^{j\omega_n}) &= \frac{2\pi^2}{j} \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \left[\delta\left(\omega_m - \frac{2\pi}{5} - 2\pi k\right) \delta\left(\omega_n - \frac{\pi}{3} - 2\pi\ell\right) \right. \\ &\quad \left. - \delta\left(\omega_m + \frac{2\pi}{5} - 2\pi k\right) \delta\left(\omega_n + \frac{\pi}{3} - 2\pi\ell\right) \right] \end{aligned}$$

ΔΙΑΚΡΙΤΟΣ ΧΡΟΝΟΣ

Example: DFT of $f(x) = \sin(2\pi x)$, $N = 8$



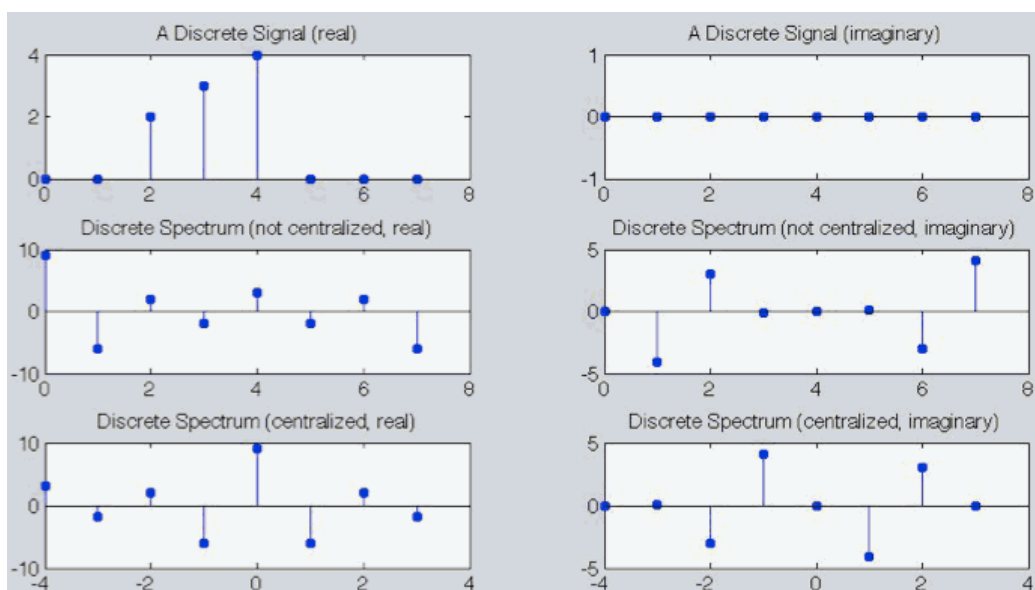
Example: 1D DFT 8-point



$$\mathbf{X}_r = [3.18, -2.16, 0.71, -0.66, 1.06, -0.66, 0.71, -2.61]^T$$

$$\mathbf{X}_j = [0.0, -1.46, 1.06, -0.04, 0.0, 0.04, -1.06, 1.46]^T$$

Example: 1D DFT 8-point



Carrying out the DFT matrix multiplication:

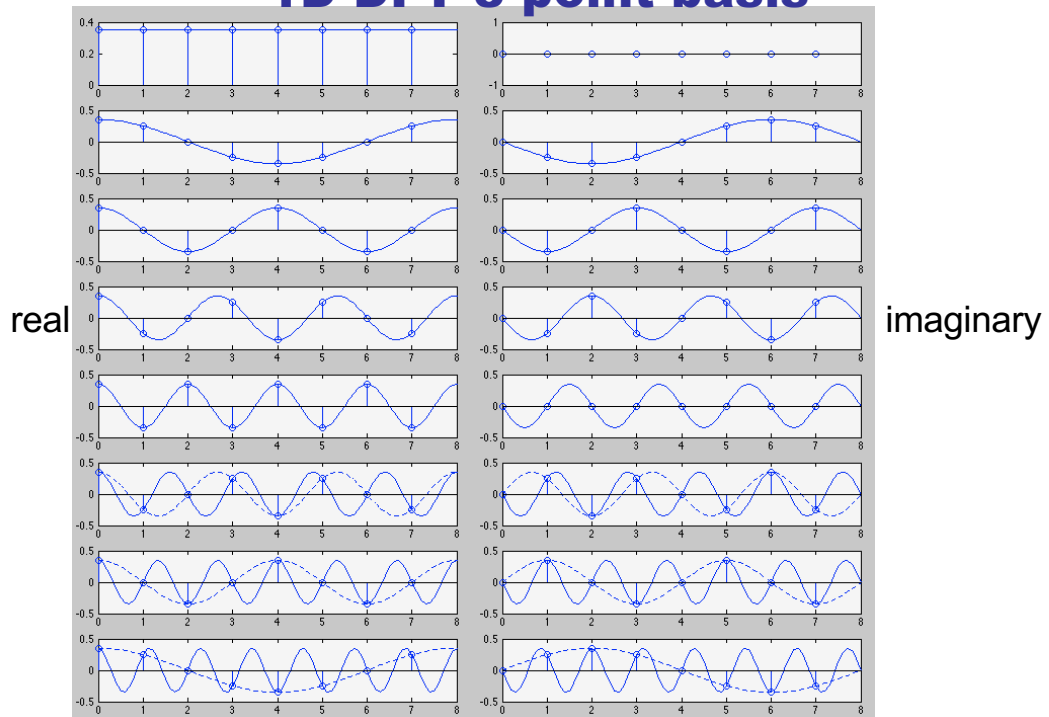
$$\mathbf{X} = \mathbf{W} \mathbf{x}$$

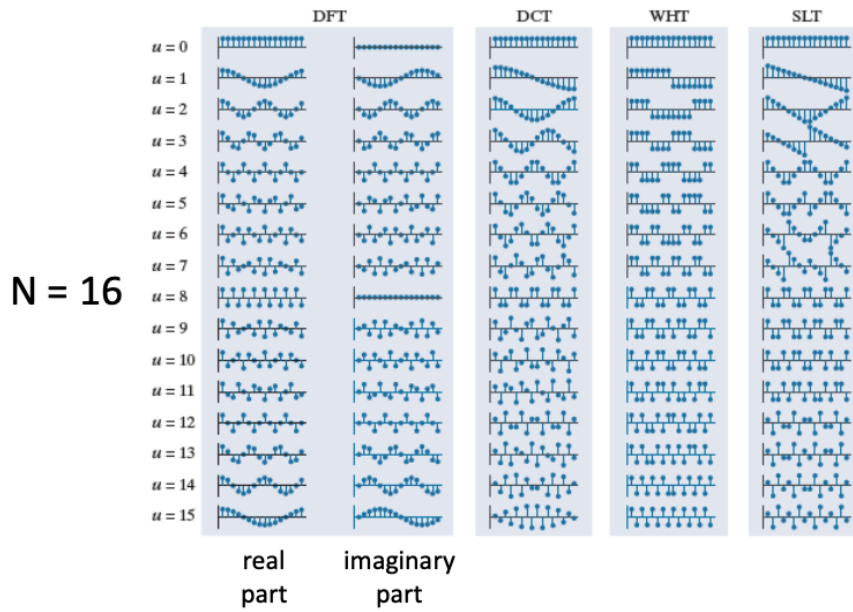
we get the $N = 8$ DFT coefficients $\mathbf{X} = \mathbf{X}_r + j\mathbf{X}_j$:

$$\mathbf{X}_r = [3.18, -2.16, 0.71, -0.66, 1.06, -0.66, 0.71, -2.61]^T$$

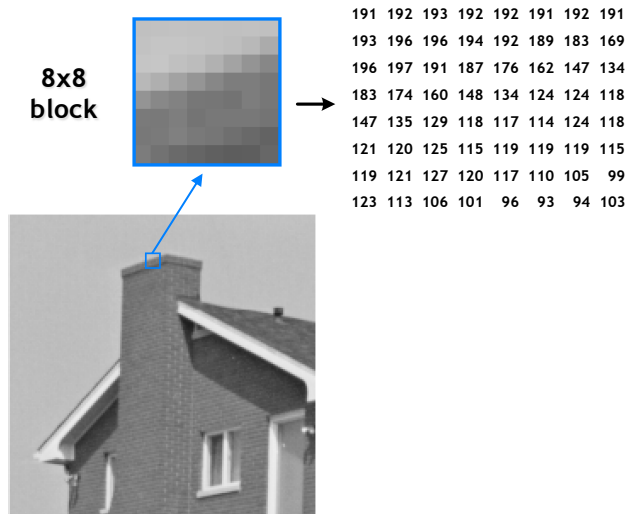
$$\mathbf{X}_j = [0.0, -1.46, 1.06, -0.04, 0.0, 0.04, -1.06, 1.46]^T$$

1D DFT 8-point basis





Grayscale Image

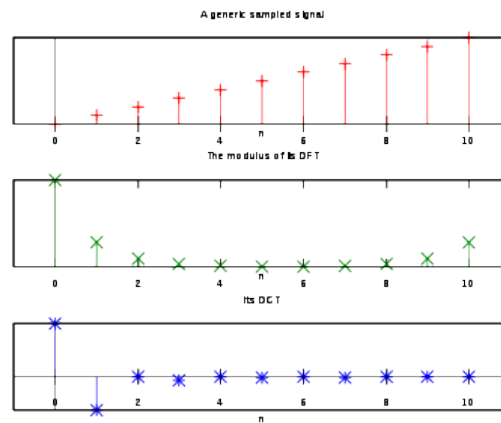


Colour Image: RGB space



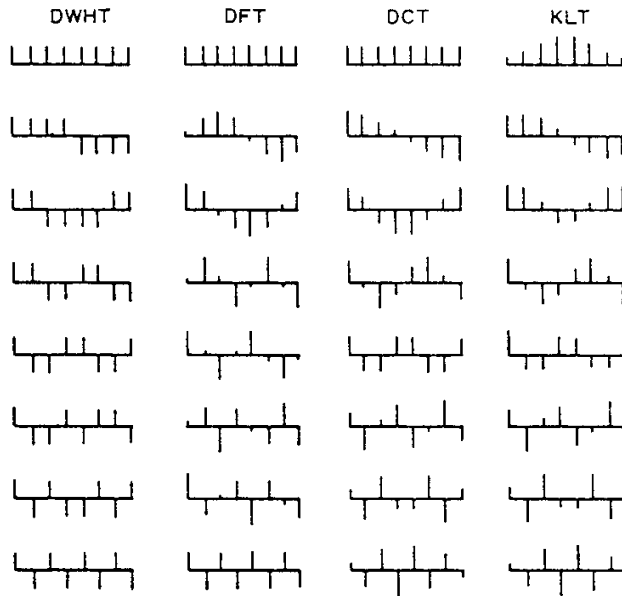
DISCRETE COSINE TRANSFORM - II

$$X_k = \sum_{n=0}^{N-1} x_n \cos \left[\frac{\pi}{N} \left(n + \frac{1}{2} \right) k \right] \quad k = 0, \dots, N-1.$$



Transform Coding

Basis vectors for 1-D transforms of order N=8



For each set of basis vectors, all vectors except the first one (first row) are zero mean sequences.

Discrete Walsh Hadamard Transform: square waves of different **sequences**.

2D DFT

$$F[k, l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n] e^{-j2\pi \left(\frac{k}{M}m + \frac{l}{N}n \right)}$$

where $k = 0, 1, \dots, M - 1$ and $l = 0, 1, \dots, N - 1$

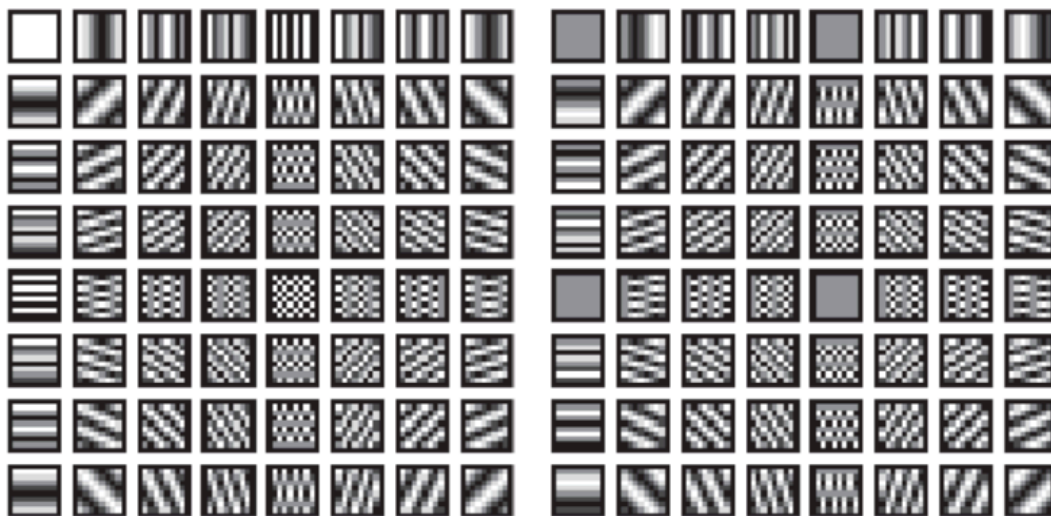
$$F[k, l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n] e^{-j2\pi \left(\frac{k}{M}m + \frac{l}{N}n \right)}$$

where $k = 0, 1, \dots, M - 1$ and $l = 0, 1, \dots, N - 1$

Inverse DFT

$$f[m, n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k, l] e^{j2\pi \left(\frac{k}{M}m + \frac{l}{N}n \right)}$$

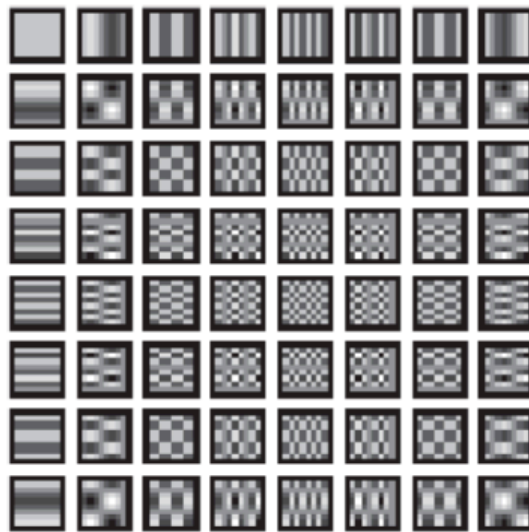
Discrete Fourier transform (DFT) basis images



real part

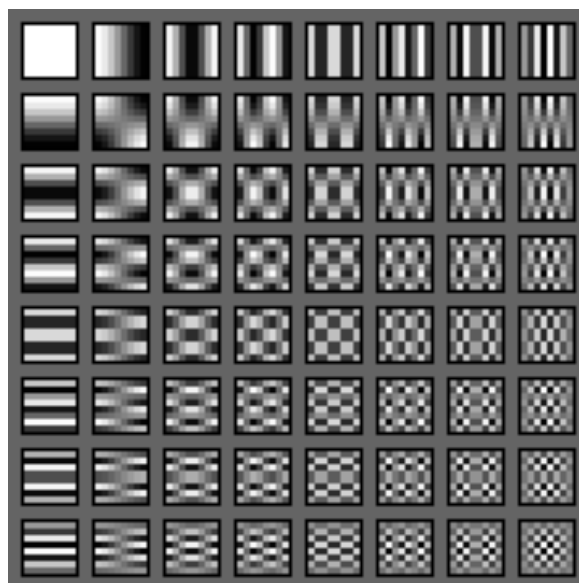
imaginary part

Discrete Hartley transform (DHT) basis images

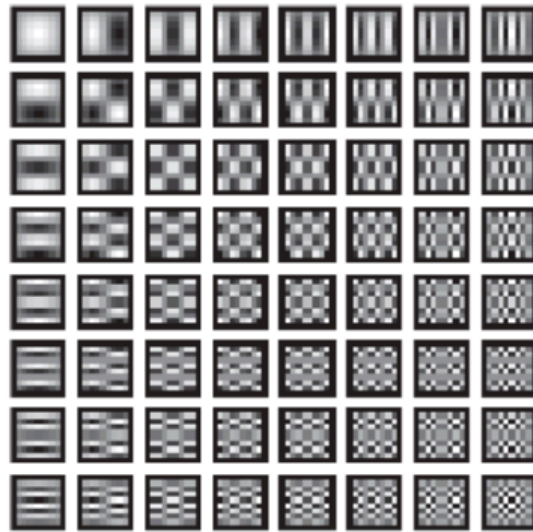


Transform Coding

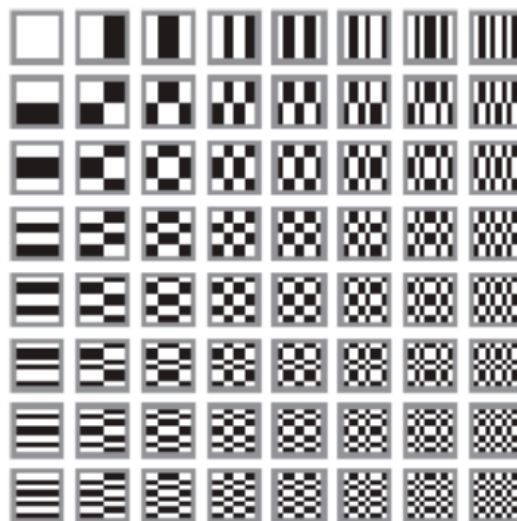
2D DCT 8x8 basis images



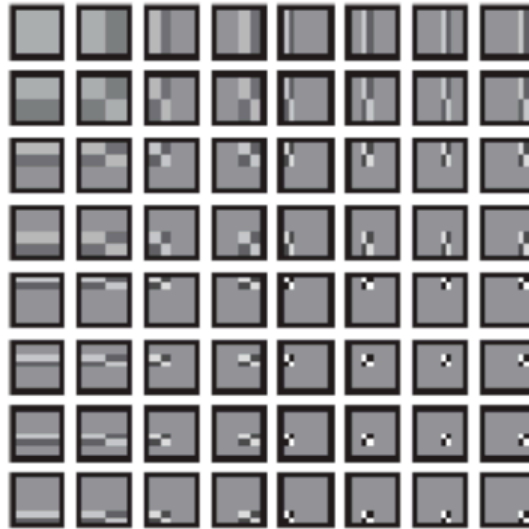
Discrete sine transform (DST) basis images



Walsh-Hadamard transform (WHT) basis images



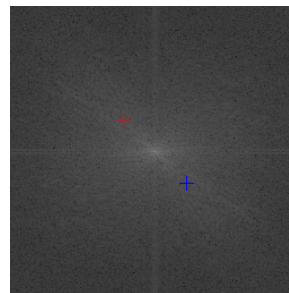
Haar transform (HAAR) basis images



Discrete Fourier Transform (DFT)

Forward DFT
$$F(u,v) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \exp\left[-j2\pi \frac{ux+vy}{N}\right]$$

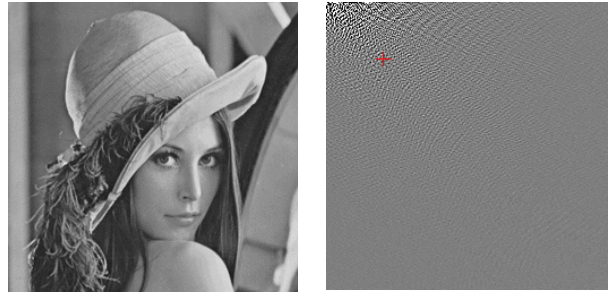
Inverse DFT
$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u,v) \exp\left[j2\pi \frac{ux+vy}{N}\right]$$



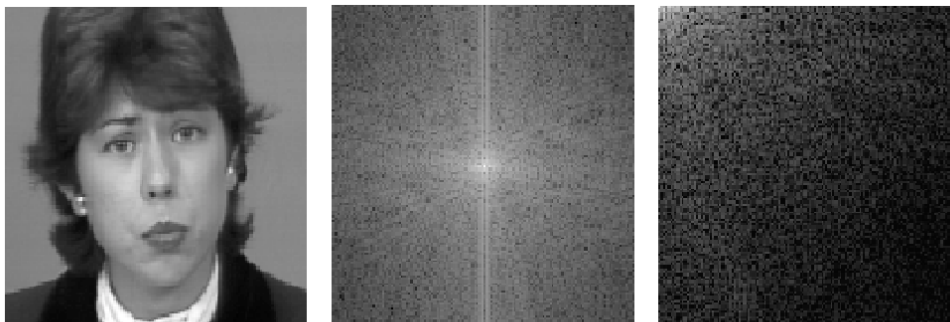
Discrete Cosine Transform (DCT)

Forward DCT
$$C(u,v) = \alpha(u)\alpha(v) \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2y+1)v}{2N}\right]$$

Inverse DCT
$$f(x,y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \alpha(u)\alpha(v) C(u,v) \cos\left[\frac{\pi(2x+1)u}{2N}\right] \cos\left[\frac{\pi(2y+1)v}{2N}\right]$$



Transform Coding

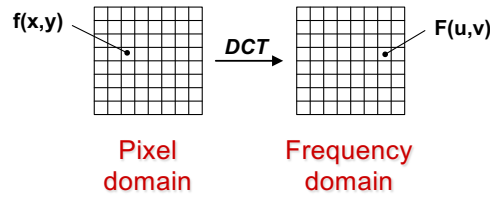


Original Image

DFT

DCT

Discrete Cosine Transform (DCT)



2D-Forward DCT (8x8 block):

$$F(u,v) = \frac{1}{4} C(u) C(v) \sum_{x=0}^7 \sum_{y=0}^7 f(x,y) * \left[\cos \frac{(2x+1) u\pi}{16} \cos \frac{(2y+1) v\pi}{16} \right]$$

where:

$$C(u), C(v) = \frac{1}{\sqrt{2}} \quad u, v = 0$$

$$C(u), C(v) = 1 \quad \text{otherwise}$$

2-D DISCRETE COSINE TRANSFORM - II

$$\begin{aligned} X_{k_1, k_2} &= \sum_{n_1=0}^{N_1-1} \left(\sum_{n_2=0}^{N_2-1} x_{n_1, n_2} \cos \left[\frac{\pi}{N_2} \left(n_2 + \frac{1}{2} \right) k_2 \right] \right) \cos \left[\frac{\pi}{N_1} \left(n_1 + \frac{1}{2} \right) k_1 \right] \\ &= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1, n_2} \cos \left[\frac{\pi}{N_1} \left(n_1 + \frac{1}{2} \right) k_1 \right] \cos \left[\frac{\pi}{N_2} \left(n_2 + \frac{1}{2} \right) k_2 \right]. \end{aligned}$$

JPEG Compression Example (3:1 or 7.8 bpp)



Original: 1,187,986 bytes or 24 bpp

407,024 bytes or 3:1 or 7.8 bpp

JPEG Compression Example (11:1 or 2 bpp)



Original: 1,187,986 bytes or 24 bpp

103,642 bytes or 11:1 or 2 bpp

JPEG Compression Example (29:1 or 0.8 bpp)



Original: 1,187,986 bytes or 24 bpp



40,920 bytes or 29:1 or 0.8 bpp

JPEG Compression Example (73:1 or 0.3 bpp)

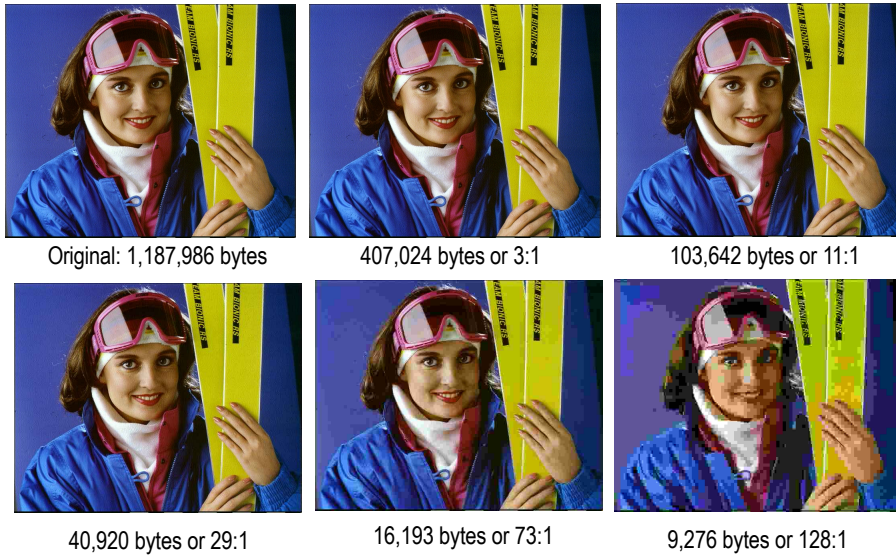


Original: 1,187,986 bytes or 24 bpp

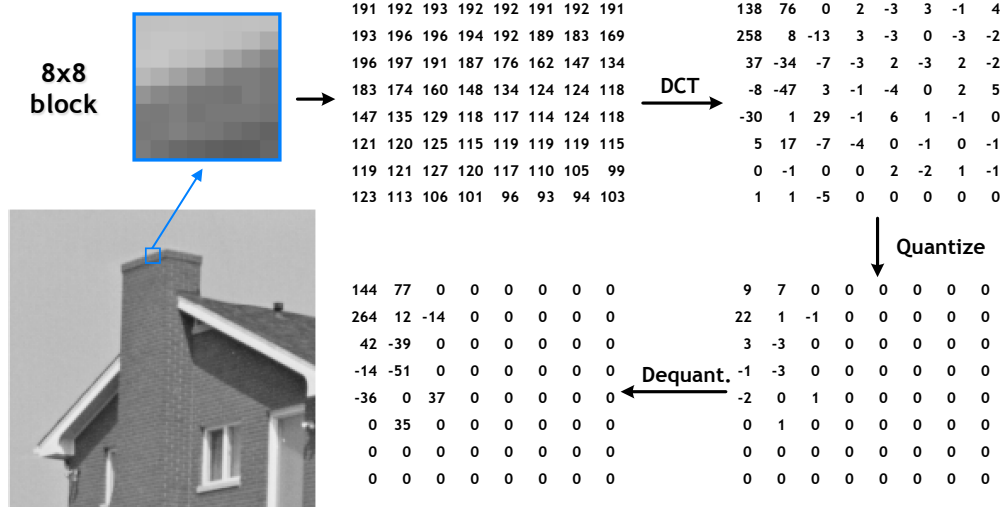


16,193 bytes or 73:1 or 0.3 bpp

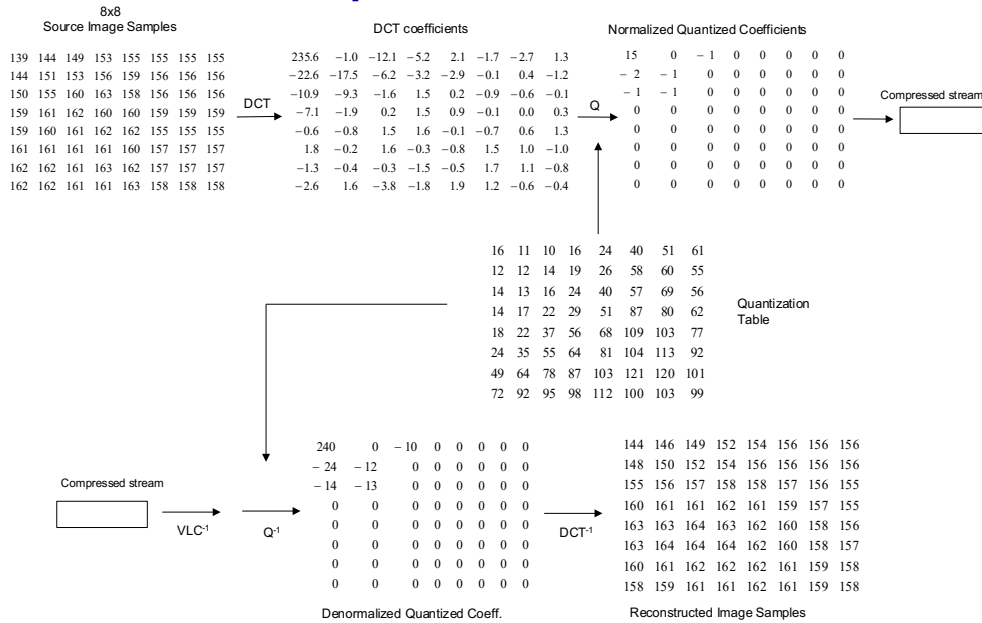
Lossy Image Compression Examples



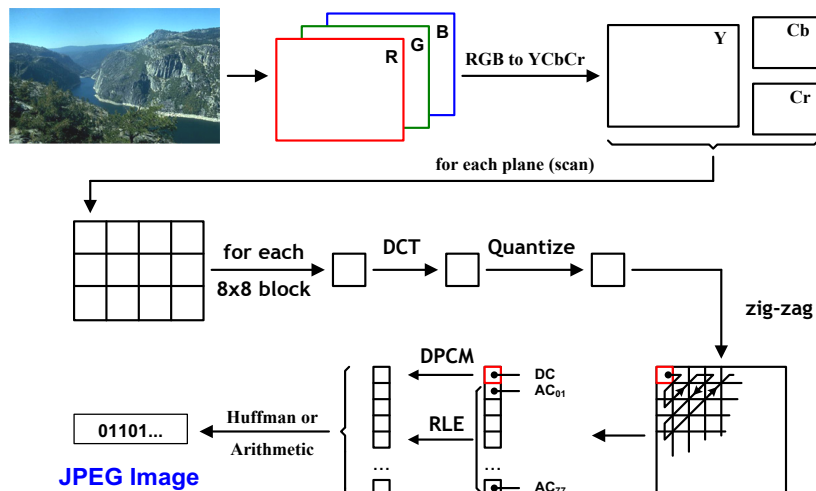
JPEG Example: 8x8 Block Encoding and Decoding



JPEG Example: 8x8 Block Encoding and Decoding



JPEG Encoder (in detail)



2-D Discrete Space Fourier Methods

A) Discrete Space FT

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-jum} e^{-jvn}$$

u: vert. freq.
v: horiz. freq.
(rads)

2-D Discrete Space Fourier Methods

A) Discrete Space FT

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-jum} e^{-jvn}$$

u: vert. freq.
v: horiz. freq.
(rads)

- properties analogous to DTFT

$$g[m,n] = h[m,n] * f[m,n] \xrightarrow{\text{DSFT}} G(u,v) = H(u,v) F(u,v)$$

convolution in space

multiplication in
frequency

2-D Discrete Fourier Transform

$$F[k, \ell] = F(u, v) \left| \begin{array}{l} u = \frac{2\pi}{N} k, v = \frac{2\pi}{N} \ell \\ k = 0, 1, \dots, N-1 \\ \ell = 0, 1, \dots, N-1 \end{array} \right. \quad \begin{array}{l} k = 0, 1, \dots, N-1 \\ \ell = 0, 1, \dots, N-1 \end{array}$$

(assumes $N \times N$ image)

2-D Discrete Fourier Transform

$$F[k, \ell] = F(u, v) \left| \begin{array}{l} u = \frac{2\pi}{N} k, v = \frac{2\pi}{N} \ell \\ k = 0, 1, \dots, N-1 \\ \ell = 0, 1, \dots, N-1 \end{array} \right.$$

(assumes $N \times N$ image)

$$F[k, \ell] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m, n] e^{-j \frac{2\pi}{N} km} e^{-j \frac{2\pi}{N} \ell n} \quad \text{2D DFT}$$

2-D Discrete Fourier Transform

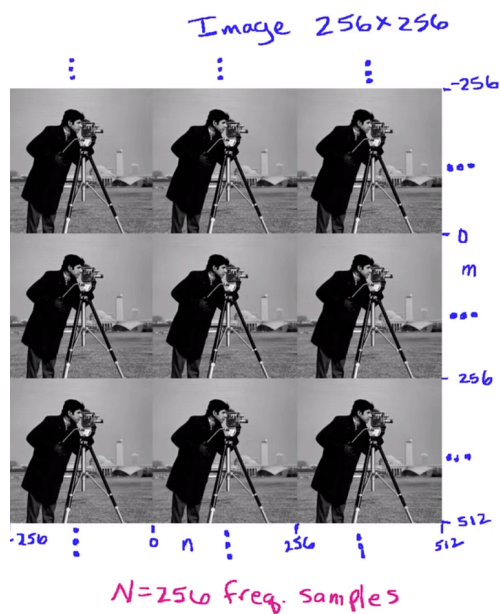
$$F[k, l] = F(u, v) \left| \begin{array}{l} u = \frac{2\pi}{N} k, v = \frac{2\pi}{N} l \\ k = 0, 1, \dots, N-1 \\ l = 0, 1, \dots, N-1 \end{array} \right.$$

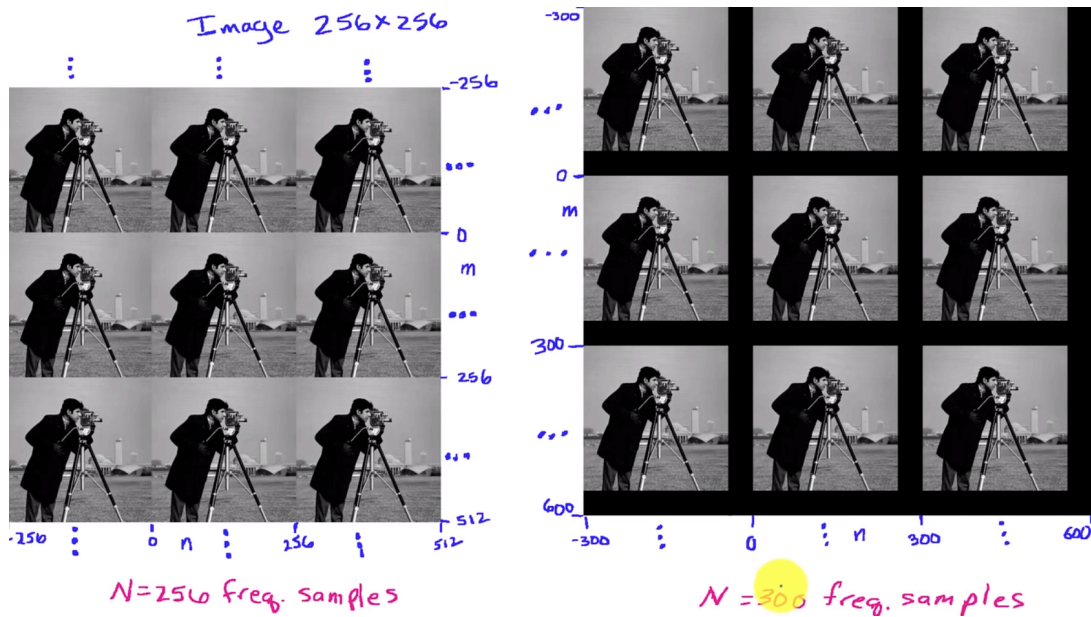
(assumes $N \times N$ image)

$$F[k, l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m, n] e^{-j \frac{2\pi}{N} km} e^{-j \frac{2\pi}{N} ln} \quad \text{2D DFT}$$

$$f[m, n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F[k, l] e^{j \frac{2\pi}{N} km} e^{j \frac{2\pi}{N} ln} \quad \text{2D Inv. DFT}$$

Sampling in frequency at $\frac{2\pi}{N}$ introduces N periodicity in $[m, n]$





DFT Convolution - Multiplication

$$\begin{array}{ccc}
 H[k_x, k_y] F[k_x, k_y] & \xleftrightarrow{2D \text{ DFT } iN} & h[m_x, n_x] \otimes f[m_x, n_y] \\
 \text{multiplication} & & \text{2D circular convolution} \\
 & & (N \text{ periodic extension due to freq. sampling})
 \end{array}$$

DFT Convolution - Multiplication

$$\begin{array}{ccc} H[k,l] F[k,l] & \xrightarrow{\text{2D DFT } iN} & h[m,n] \circledast f[m,n] \\ \text{multiplication} & & \text{2D circular convolution} \\ & & \text{(N periodic extension due to freq. sampling)} \end{array}$$

Obtain linear convolution with zero padding!

if h is $M \times M$, f is $P \times P$, choose $N \geq M+P-1$

$$\text{then } h[m,n] * f[m,n] = h[m,n] \circledast f[m,n]$$

zero pad to $N \geq M+P-1$
2D DFTs
Multiply DFT coeffs
2D Inv DFT

Computing the 2D DFT

$$F[k,l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m,n] e^{-j \frac{2\pi}{N} km} e^{-j \frac{2\pi}{N} ln}$$

Computing the 2D DFT

$$\begin{aligned}
 F[k, l] &= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f[m, n] e^{-j \frac{2\pi}{N} km} e^{-j \frac{2\pi}{N} ln} \\
 &= \sum_{m=0}^{N-1} \underbrace{\left(\sum_{n=0}^{N-1} f[m, n] e^{-j \frac{2\pi}{N} ln} \right)}_{\text{1-D DFT over } n} e^{-j \frac{2\pi}{N} km} \quad \leftarrow \begin{array}{l} \text{use 1-D FFT for each } m \\ N \text{ values of } m \\ O(N^2 \log_2 N) \end{array} \\
 &= \sum_{m=0}^{N-1} \tilde{f}[m, l] e^{-j \frac{2\pi}{N} km} \quad \leftarrow \begin{array}{l} \text{1-D FFT for each value } l \\ N \text{ values of } l \\ > O(N^2 \log_2 N) \end{array}
 \end{aligned}$$

Overall complexity $\sim O(N^2 \log_2 N)$
 \uparrow total # pixels

2D Discrete Fourier Transform

$$x[m, n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X[k, l] \underbrace{S_{k, l}[m, n]}_{\substack{\text{weights} \\ \text{2D complex sinusoids} \\ \text{frequency } \frac{k}{N}, \frac{l}{N}}}$$

2D Discrete Fourier Transform

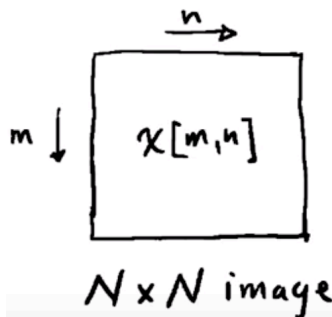
$$x[m, n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \underbrace{X[k, l]}_{\text{weights}} \underbrace{s_{k, l}[m, n]}_{\text{2D complex sinusoids}} \quad \text{frequency } \frac{k}{N}, \frac{l}{N}$$

$$x[m, n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X[k, l] e^{j2\pi(\frac{k}{N})m} e^{j2\pi(\frac{l}{N})n}$$

DFT
coeffs

$$X[k, l] = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x[m, n] e^{-j2\pi\frac{k}{N}m} e^{-j2\pi\frac{l}{N}n}$$

Two dimensional complex sinusoids

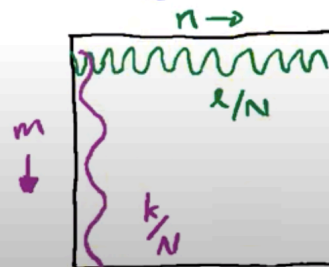


$$\text{Define } s_{k, l}[m, n] = e^{j2\pi(\frac{k}{N})m} e^{j2\pi(\frac{l}{N})n}$$

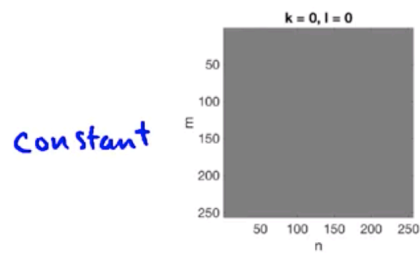
vertical complex sinusoid w. frequency $\frac{k}{N}$ horizontal complex sinusoid w. frequency $\frac{l}{N}$

horizontal freq $\frac{l}{N}$ cycles/sample

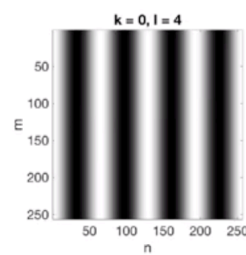
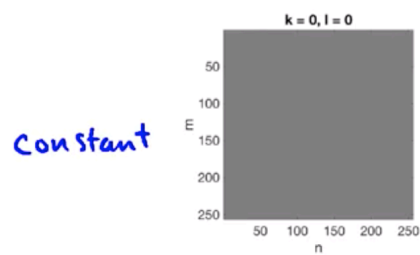
vertical freq $\frac{k}{N}$ cycles/sample



Consider $\text{real} \left\{ e^{j2\pi \left(\frac{k}{N}\right) m} e^{j2\pi \left(\frac{l}{N}\right) n} \right\}$

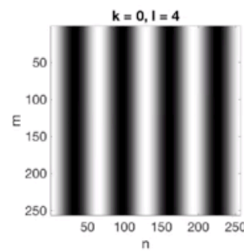
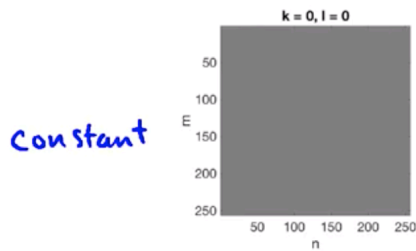


Consider $\text{real} \left\{ e^{j2\pi \left(\frac{k}{N}\right) m} e^{j2\pi \left(\frac{l}{N}\right) n} \right\}$

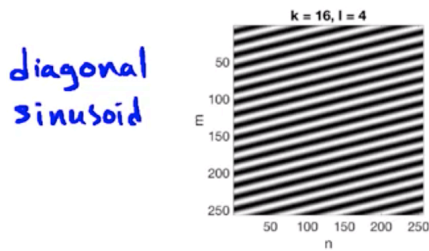


horizontal
sinusoid

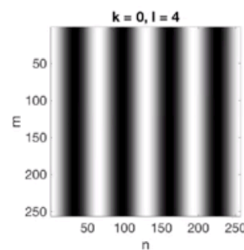
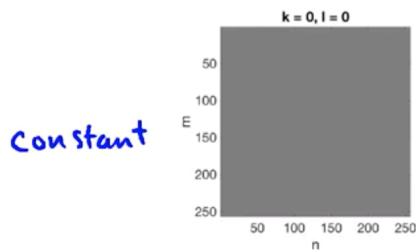
Consider $\text{real} \left\{ e^{j2\pi \left(\frac{k}{N}\right)m} e^{j2\pi \left(\frac{l}{N}\right)n} \right\}$



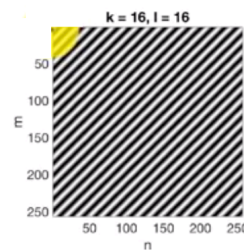
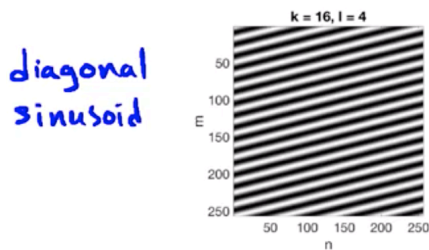
horizontal
sinusoid



Consider $\text{real} \left\{ e^{j2\pi \left(\frac{k}{N}\right)m} e^{j2\pi \left(\frac{l}{N}\right)n} \right\}$

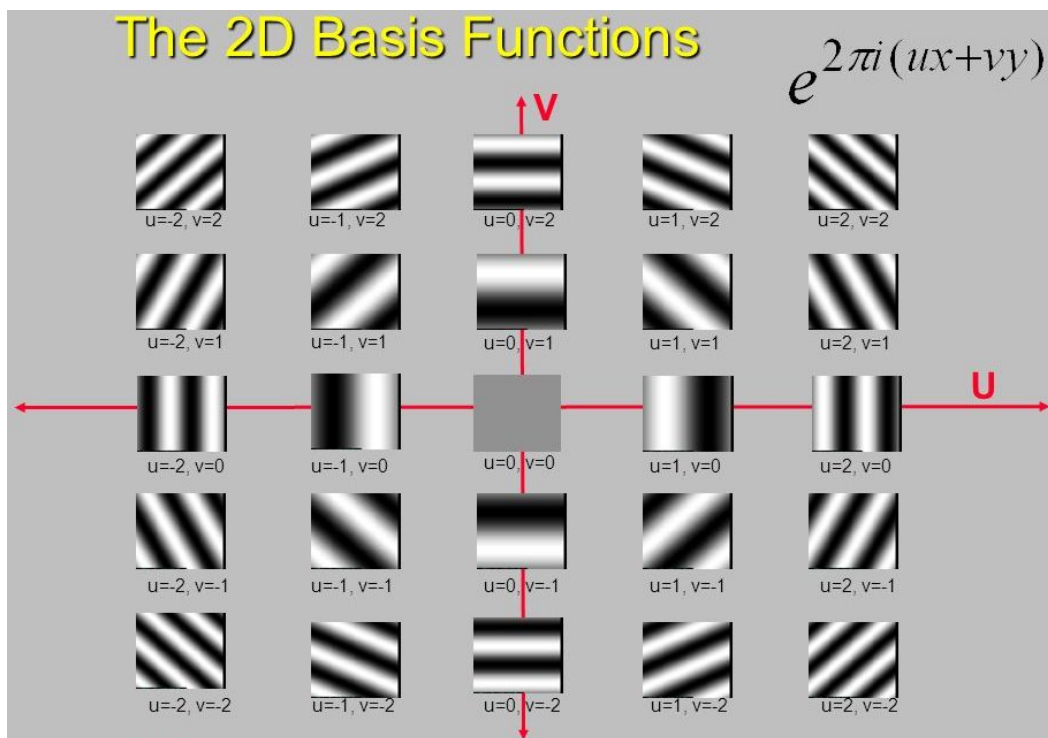
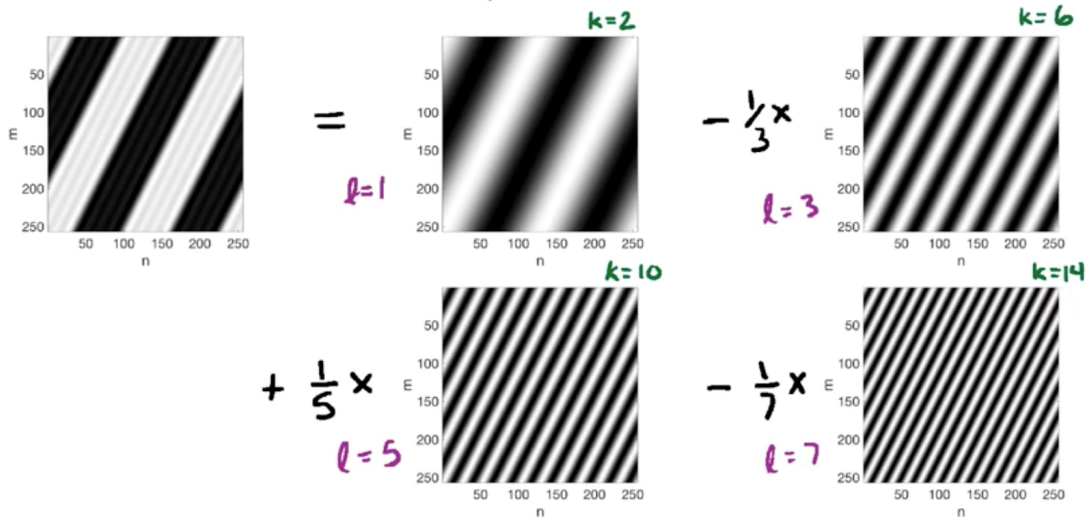


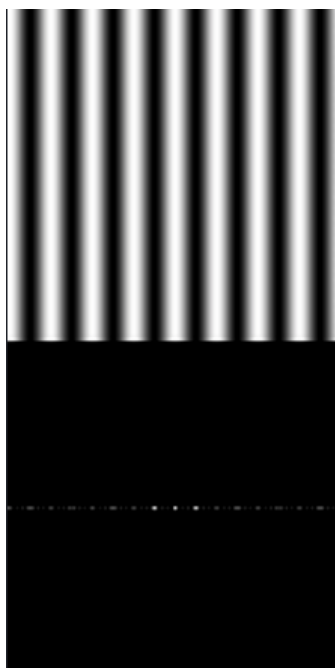
horizontal
sinusoid



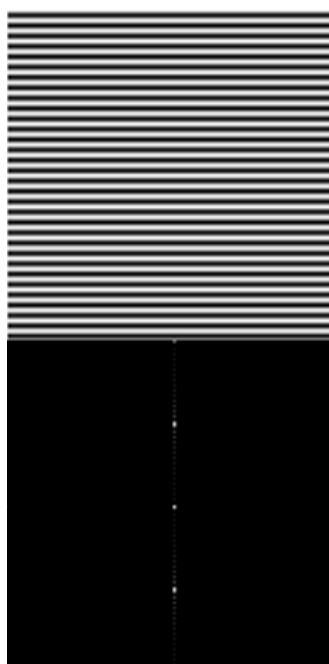
diagonal
sinusoid

$$\text{Example: } x[m,n] = \sum_{p=1}^4 \frac{(-1)^{p-1}}{2^{p-1}} \text{real} \left(e^{j2\pi \frac{(2p-1)}{N} m} e^{j4\pi \frac{(2p-1)}{N} n} \right)$$

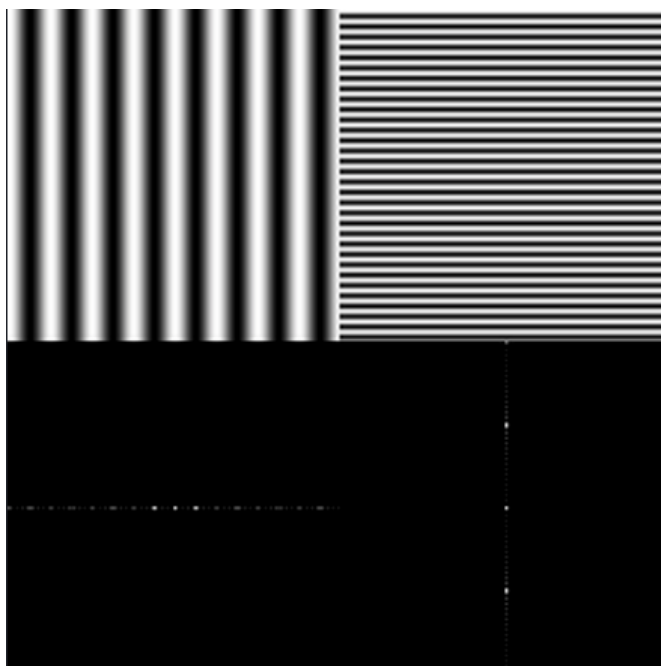




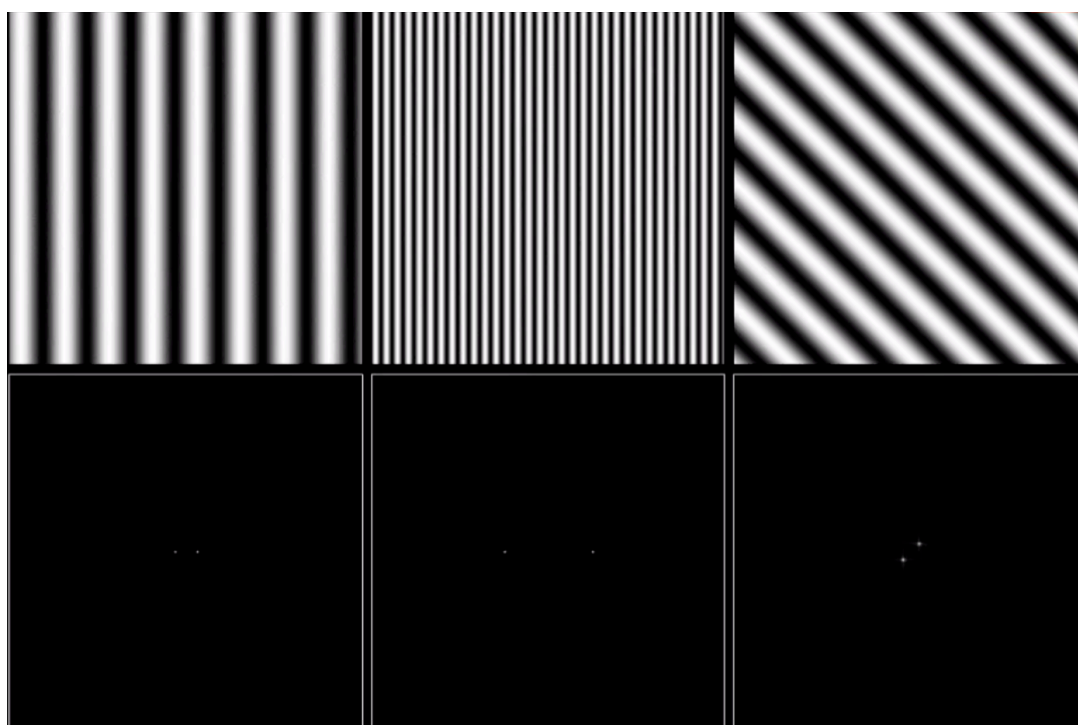
<https://www.youtube.com/watch?v=oACegp4iGi0>



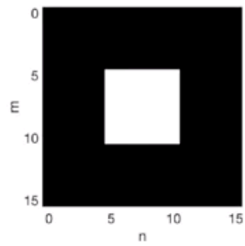
<https://www.youtube.com/watch?v=oACegp4iGi0>



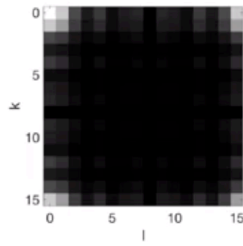
<https://www.youtube.com/watch?v=oACegp4iGi0>



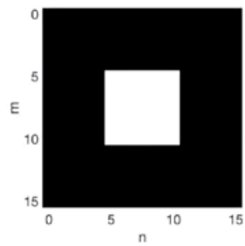
Examples: $x[m,n]$



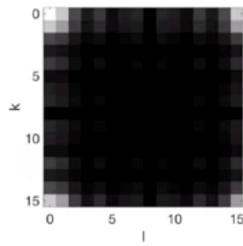
$|X[k,l]|$



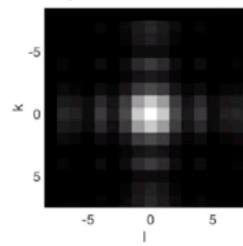
Examples: $x[m,n]$



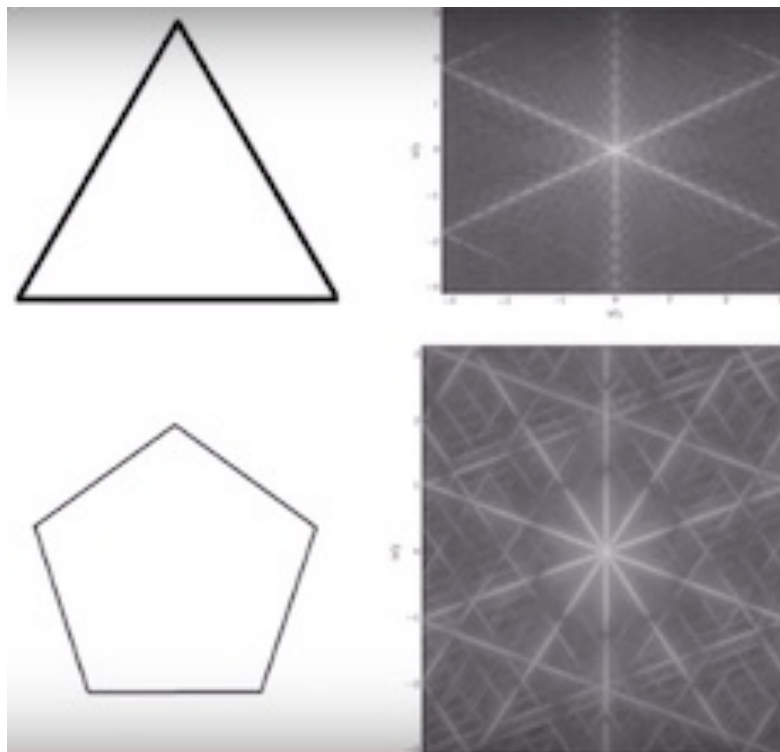
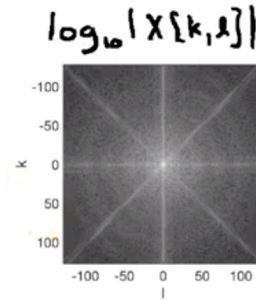
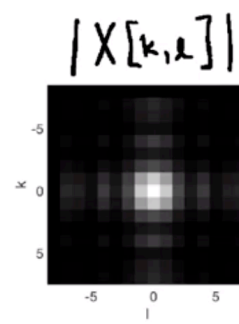
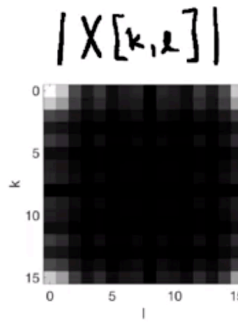
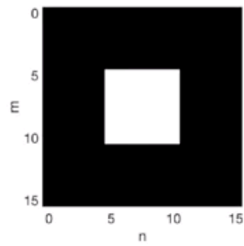
$|X[k,l]|$



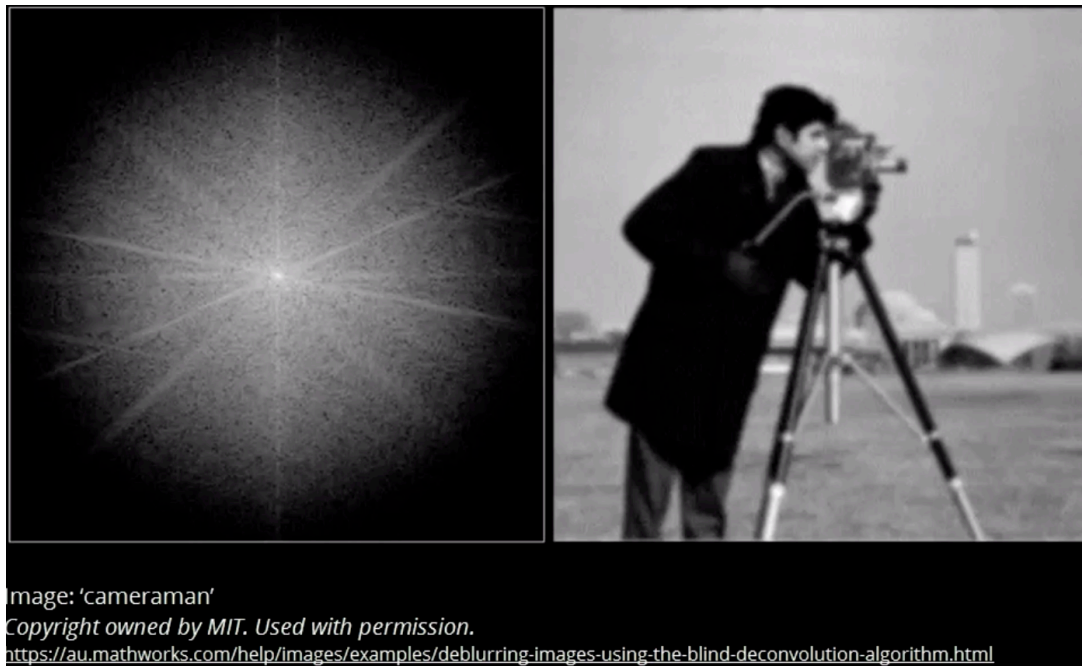
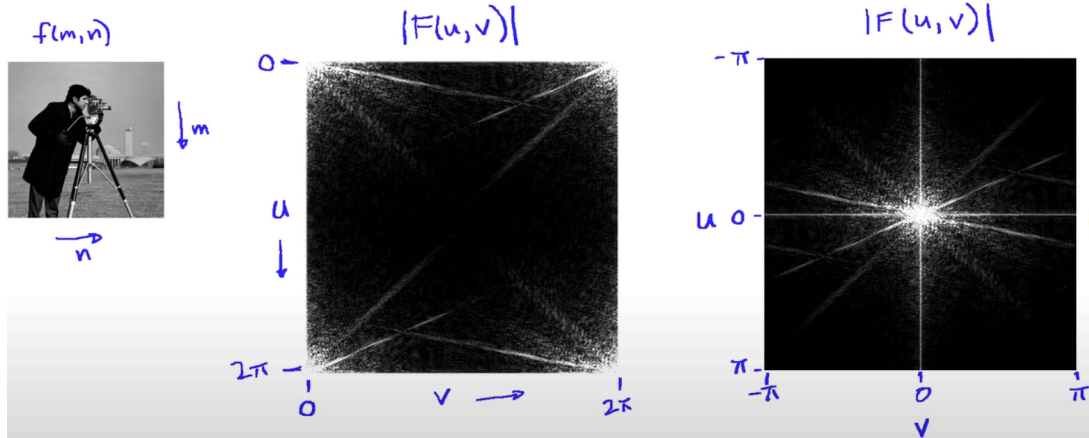
$|X[k,l]|$

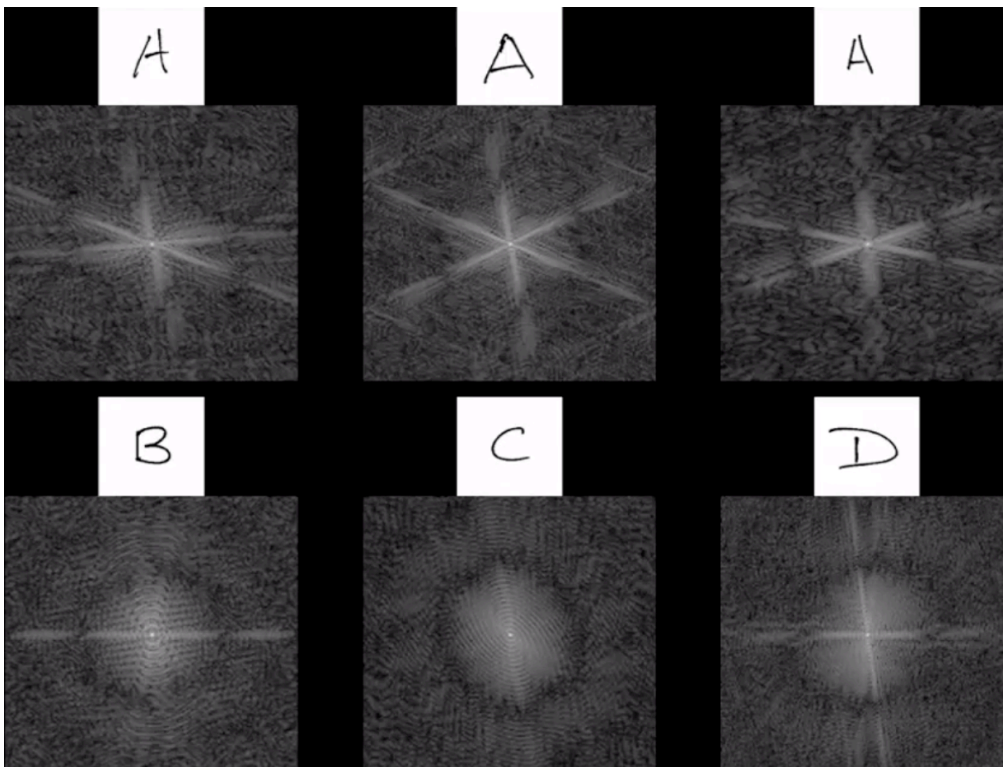
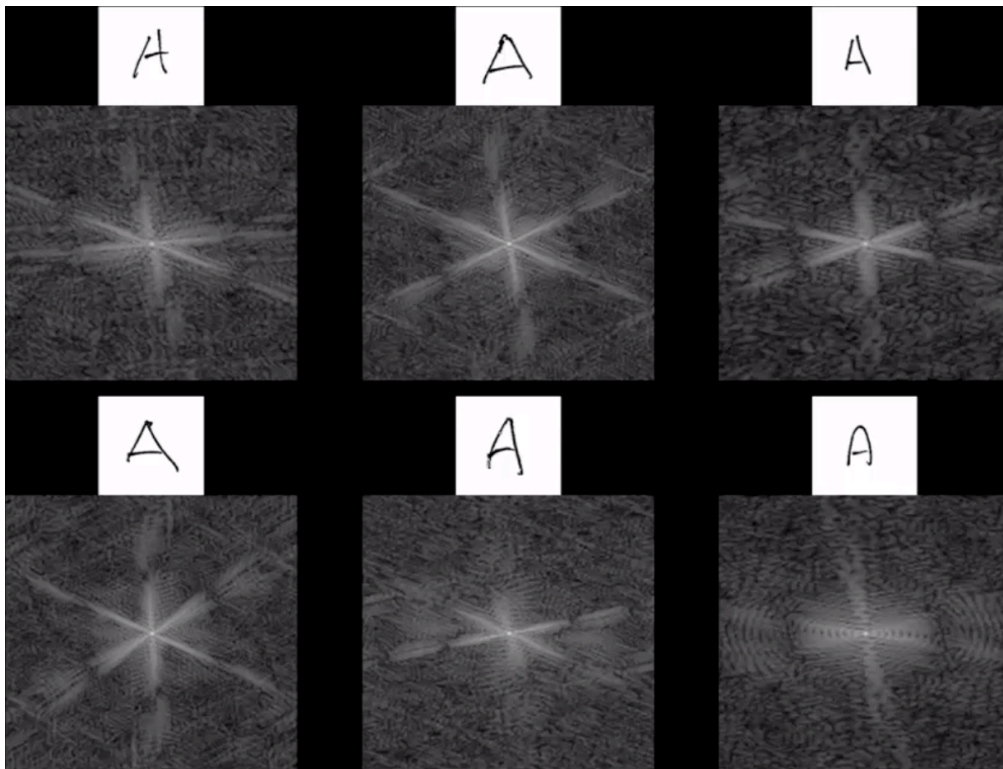


Examples: $x[m,n]$



Example: Cameraman Image







https://www.youtube.com/watch?v=aiKrrGR57al&ab_channel=drheaddamage
<https://lookingatnothing.com/>

ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΣ HARTLEY

Μετασχηματισμός Hartley

$$f(t) \xleftrightarrow{H} F_H(\Omega)$$

Ευθύς:
$$F_H(\Omega) = \int_{-\infty}^{\infty} f(t) \cos(\Omega t) dt$$

Αντίστροφος:
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_H(\Omega) \cos(\Omega t) d\Omega$$

$$\text{όπου } \cos(\Omega t) = \cos(\Omega t) + \sin(\Omega t)$$

Ο μετασχηματισμός αυτός προτάθηκε από τον Hartley το 1942.

Πρόκειται για έναν πραγματικό μετασχηματισμό (όχι μιγαδικό), ο οποίος διαφέρει από τον μετασχηματισμό Fourier κατά τη χρήση του κοσίνου αντί του συντελεστή του $\sin(\Omega t)$.

Έστω
$$F_H(\Omega) = F_{He}(\Omega) + F_{Ho}(\Omega)$$

όπου $F_{He}(\Omega)$, $F_{Ho}(\Omega)$ το άρτιο και περιττό μέρος του πραγματ. Hartley.

Άρα

$$F_{He} = \frac{F_H(\Omega) + F_H(-\Omega)}{2} = \int_{-\infty}^{\infty} f(t) \cos(\Omega t) dt = \operatorname{Re}\{F(\Omega)\}$$

$$F_{Ho} = \frac{F_H(\Omega) - F_H(-\Omega)}{2} = \int_{-\infty}^{\infty} f(t) \sin(\Omega t) dt = -\operatorname{Im}\{F(\Omega)\}$$

όπου
$$F(\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} f(t) (\cos \Omega t - j \sin \Omega t) dt$$
 ο μιγαδικός Fourier του $f(t)$.

Από τις παραπάνω σχέσεις προκύπτει:

$$F(\Omega) = F_{He}(\Omega) - j F_{Ho}(\Omega)$$

$$F_H(\Omega) = \operatorname{Re}\{F(\Omega)\} - \operatorname{Im}\{F(\Omega)\}$$

Ιδιότητες του μετασχηματισμού Hartley

Γραμμικότητα: $a f_1(t) + b f_2(t) \xrightarrow{H} a F_{H1}(\omega) + b F_{H2}(\omega)$

Κλίση στον χρόνο: $f(at) \xrightarrow{H} \frac{1}{|a|} F_H\left(\frac{\omega}{a}\right)$

Ολισθήση στο χρόνο: $f(t-t_0) \xrightarrow{H} \cos \omega t_0 F_H(\omega) + \sin \omega t_0 F_H(-\omega)$

Διαφύλαξη: $f(t) \cos \omega_0 t \xrightarrow{H} \frac{1}{2} [F_H(\omega - \omega_0) + F_H(\omega + \omega_0)]$

ΑΣΚΗΣΗ Να υπολογιστεί ο μετασχηματισμός Hartley του σήματος $f(t) = \delta(t-t_0)$.

ΛΥΣΗ

$$\begin{aligned} F_H(\omega) &= \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt = \\ &= \int_{-\infty}^{\infty} \delta(t-t_0) \cos(\omega t) dt = \\ &= \cos(\omega t_0) \end{aligned}$$

Άσκηση Να υπολογιστεί ο φτωχ. Hartley του σήματος $f(t) = e^{-t} u(t)$

Λύση Γνωρίζουμε ότι ο MF του σήματος $f(t)$ είναι:

$$F(\omega) = \frac{1}{1+j\omega} = \frac{(1-j\omega)}{(1+j\omega)(1-j\omega)} = \frac{1}{1+\omega^2} + j \frac{-\omega}{1+\omega^2}$$

$\underbrace{\hspace{10em}}_{\text{Re}\{F(\omega)\}} \quad \underbrace{\hspace{10em}}_{\text{Im}\{F(\omega)\}}$

Ειδικά ότι

$$\begin{aligned} F_H(\omega) &= \text{Re}\{F(\omega)\} - \text{Im}\{F(\omega)\} = \\ &= \frac{1}{1+\omega^2} + \frac{\omega}{1+\omega^2} = \\ &= \frac{1+\omega}{1+\omega^2} \end{aligned}$$

Άσκηση Να υπολογιστεί ο φτωχ. Hartley του τετραγωνικού παλμού $p(t)$ εύρους T , δηλ. $p(t) = \begin{cases} 1, & |t| < \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$

Λύση

$$\begin{aligned} P_H(\omega) &= \int_{-\infty}^{\infty} p(t) \cos(\omega t) dt = \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\omega t) dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} (\cos \omega t + \sin \omega t) dt = \\ &= \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos \omega t dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \omega t dt = \\ &= \frac{1}{\omega} \sin \omega t \Big|_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{1}{\omega} \left[\sin \frac{\omega T}{2} - \sin \frac{-\omega T}{2} \right] = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Omega} \left[\sin \frac{\Omega T}{2} + \sin \frac{\Omega T}{2} \right] = \\
&= \frac{2}{\Omega} \sin \frac{\Omega T}{2} = \frac{\sin \frac{\Omega T}{2}}{\frac{\Omega}{2}} = T \frac{\sin \frac{\Omega T}{2}}{\frac{\Omega T}{2}} = T \cdot \text{sinc} \left(\frac{\Omega T}{2\pi} \right) \\
&\text{όπου } \text{sinc} x = \frac{\sin \pi x}{\pi x}
\end{aligned}$$

Παρατήρηση: Το κριτήριο αυτό είναι ίδιο με εκείνο του MF της $p(t)$. Αυτό ήταν αναμενόμενο, αφού το φανταστικό μέρος του MF της $p(t)$ είναι 0 (φυσικά) και άρα ο φασικός Hartley θα ισούται με:

$$P_u(\Omega) = \text{Re}\{F(\Omega)\} - \text{Im}\{F(\Omega)\} \stackrel{\rightarrow 0}{=} \text{Re}\{F(\Omega)\}$$

HARTLEY TRANSFORM (HT)

$$H(\omega) = \{\mathcal{H}f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \text{cas}(\omega t) dt,$$

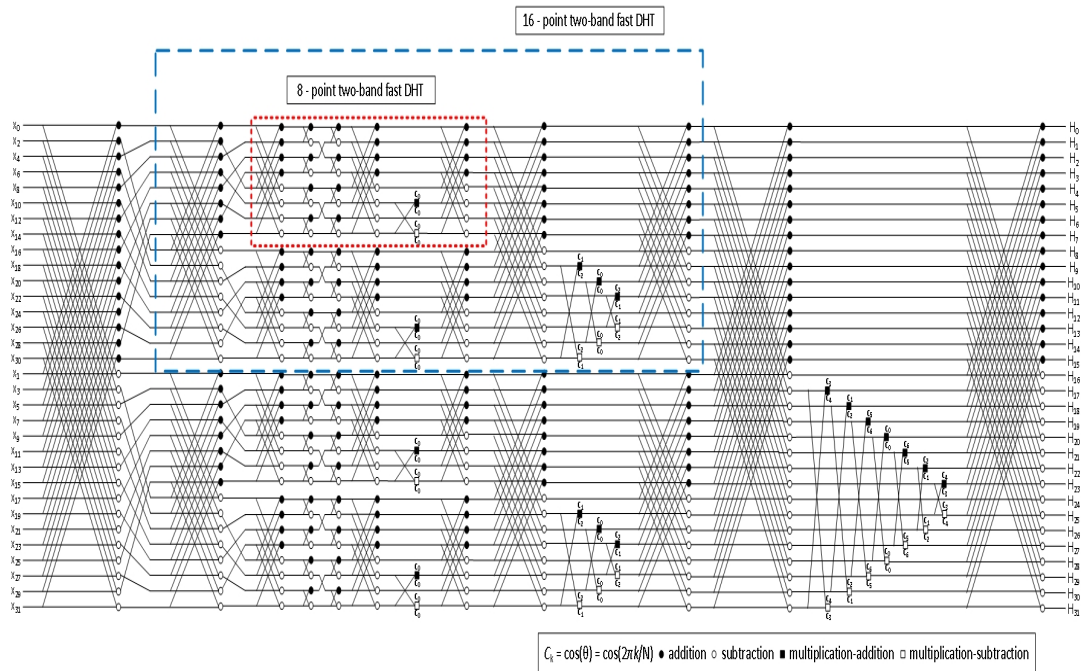
$$\text{cas}(t) = \cos(t) + \sin(t) = \sqrt{2} \sin(t + \pi/4) = \sqrt{2} \cos(t - \pi/4)$$

$$f = \{\mathcal{H}\{\mathcal{H}f\}\}.$$

$$F(\omega) = \frac{H(\omega) + H(-\omega)}{2} - i \frac{H(\omega) - H(-\omega)}{2}.$$

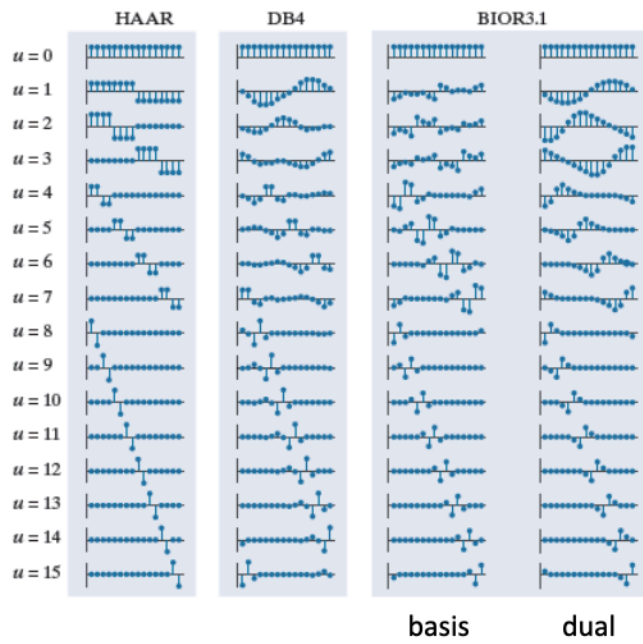
DISCRETE HARTLEY TRANSFORM (DHT)

$$H_k = \sum_{n=0}^{N-1} x_n \left[\cos\left(\frac{2\pi}{N}nk\right) + \sin\left(\frac{2\pi}{N}nk\right) \right] \quad k = 0, \dots, N-1$$

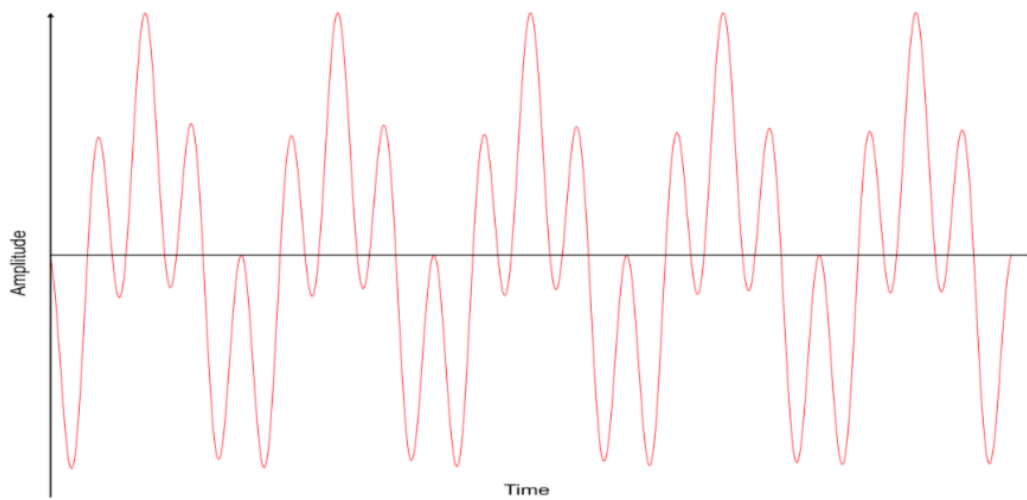


WAVELETS

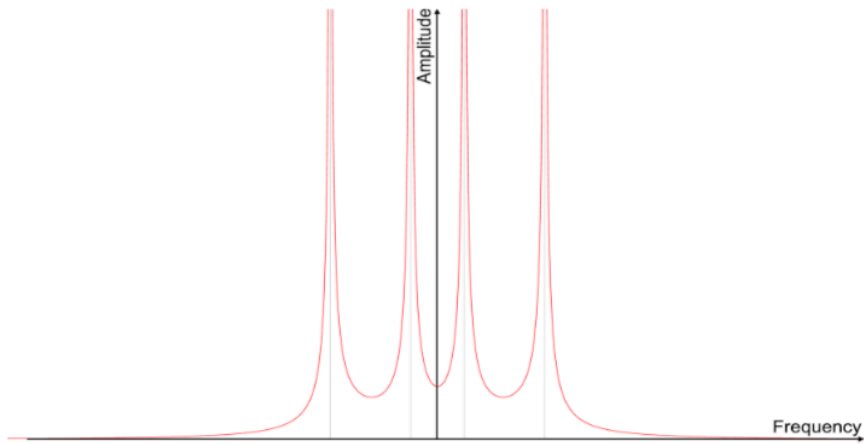
N = 16



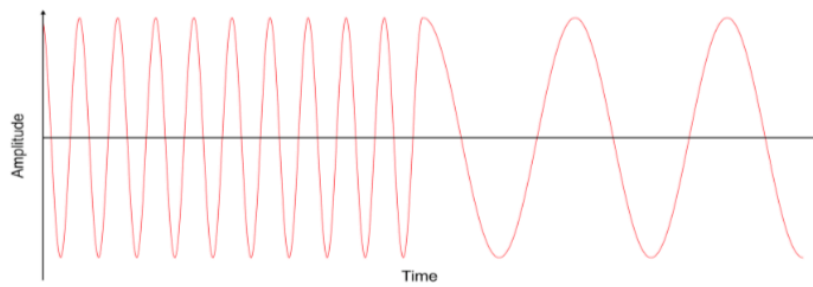
Γιατί Wavelets ?



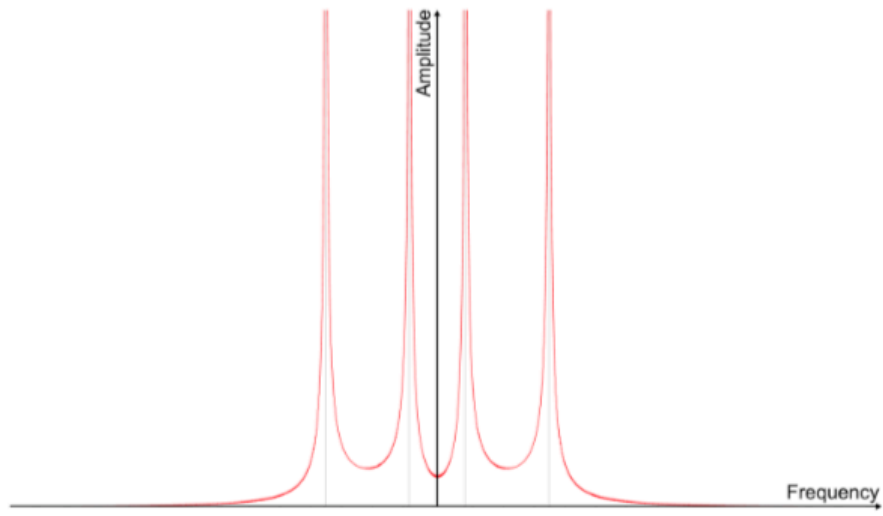
Stationary signal consisting of a sum of two sines



The Fourier Transform of the signal reveals its frequency content, consisting of two distinct frequencies



Non-stationary signal consisting of first a high-frequency sine and then a lower frequency sine



Source - <https://youtu.be/jnxqHcObNK4?t=334>

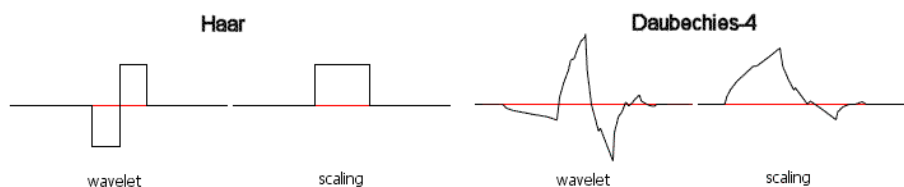
SHORT-TIME FOURIER TRANSFORM (STFT)

Continuous-time $\text{STFT}\{x(t)\}(\tau, \omega) \equiv X(\tau, \omega) = \int_{-\infty}^{\infty} x(t)w(t - \tau)e^{-j\omega t} dt$

Discrete-time $\text{STFT}\{x[n]\}(m, \omega) \equiv X(m, \omega) = \sum_{n=-\infty}^{\infty} x[n]w[n - m]e^{-j\omega n}$

WAVELET TRANSFORM (WT)

$$CWT_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t)\psi\left(\frac{t - \tau}{s}\right) dt$$



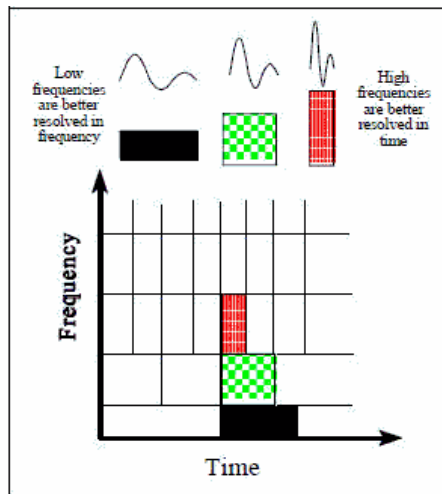
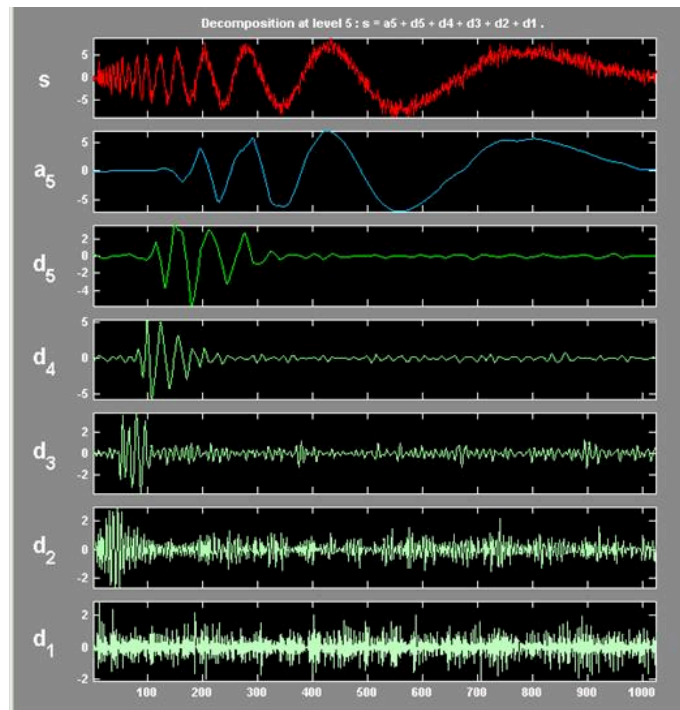
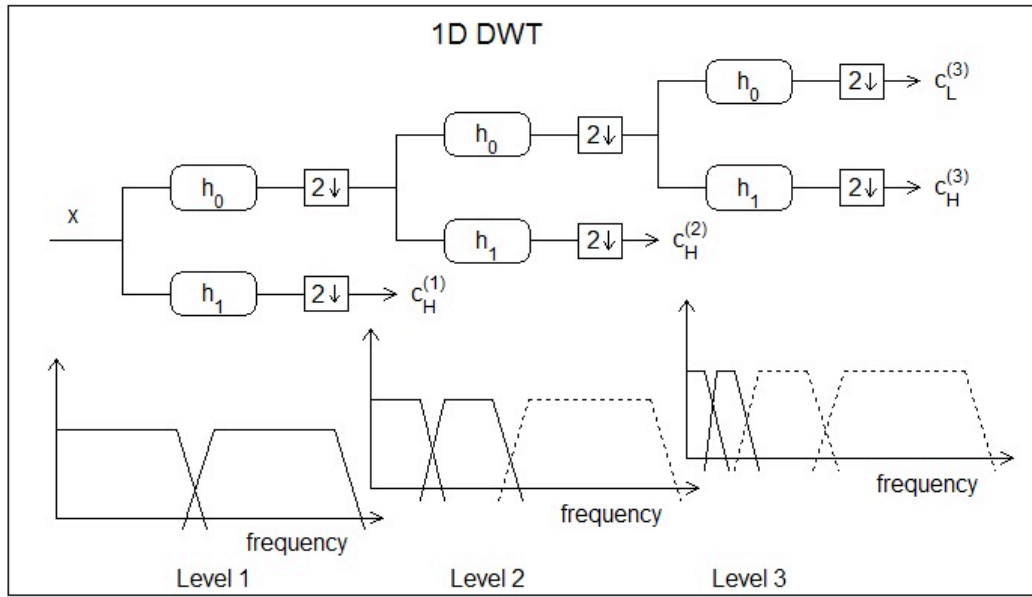
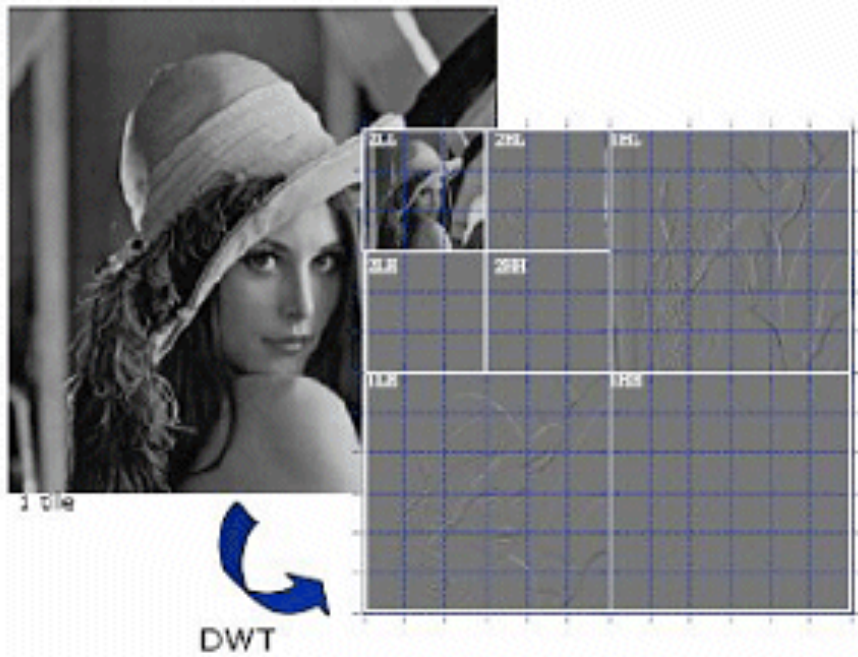


Figure 6 Wavelet Transform

DISCRETE WAVELET TRANSFORM (DWT)

$$\begin{aligned}\phi_{i,j}(k) &= 2^{i/2}h_i(k - 2^i j) \\ \psi_{i,j}(k) &= 2^{i/2}g_i(k - 2^i j)\end{aligned}$$





JPEG at 0.125 bpp



JPEG2000 at 0.125 bpp



JPEG at 0.25 bpp



JPEG2000 at 0.25 bpp



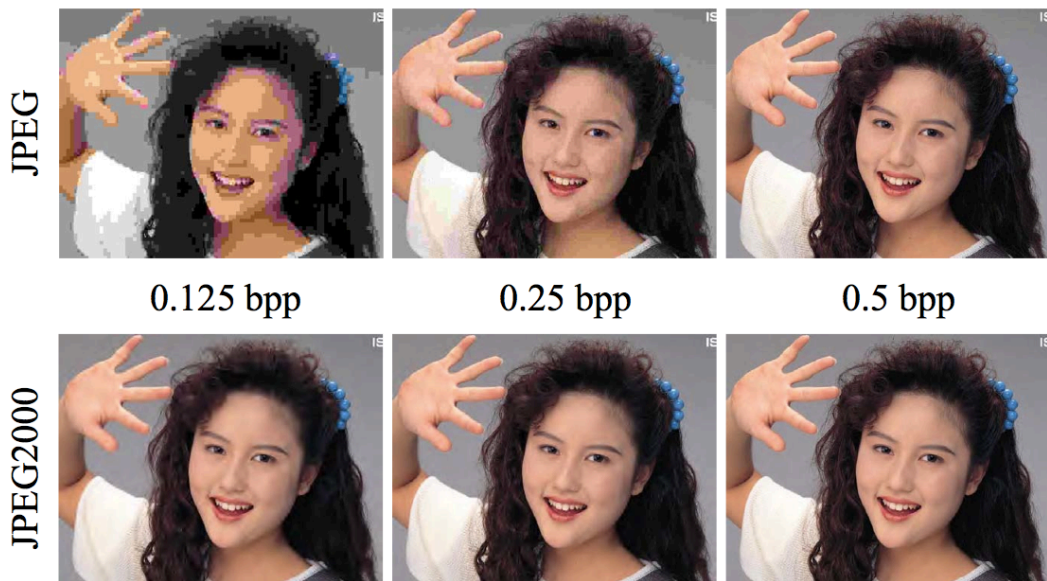
JPEG at 0.5 bpp



JPEG2000 at 0.5 bpp



JPEG vs. JPEG2000



ΠΗΓΕΣ

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5. https://en.wikipedia.org/wiki/Discrete_cosine_transform#/media/File:DCT-8x8.png
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