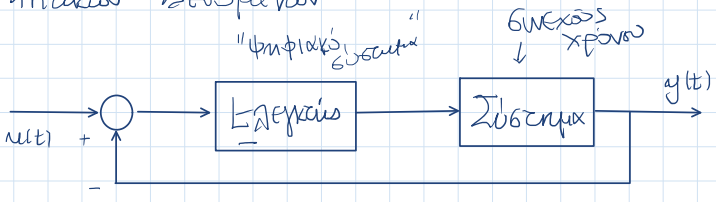
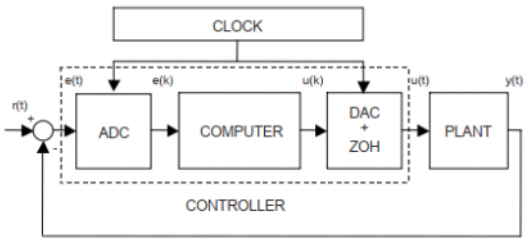
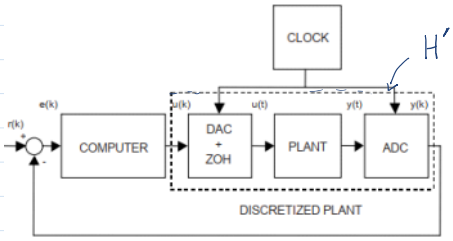
ΔΕΙΓΜΑΤΟΛΗΨΙΑ - ΑΝΑΚΑΤΑΣΚΕΥΗ

Συστήματα "Δειγματοληπτικών Δεδομένων"

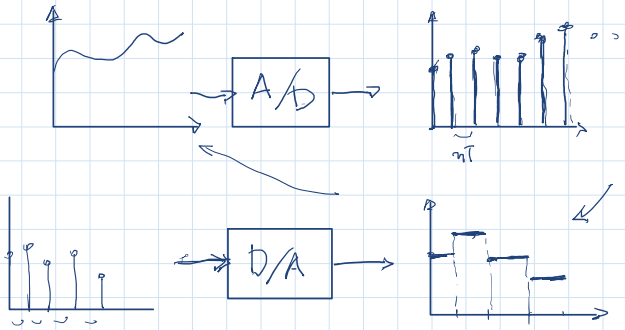






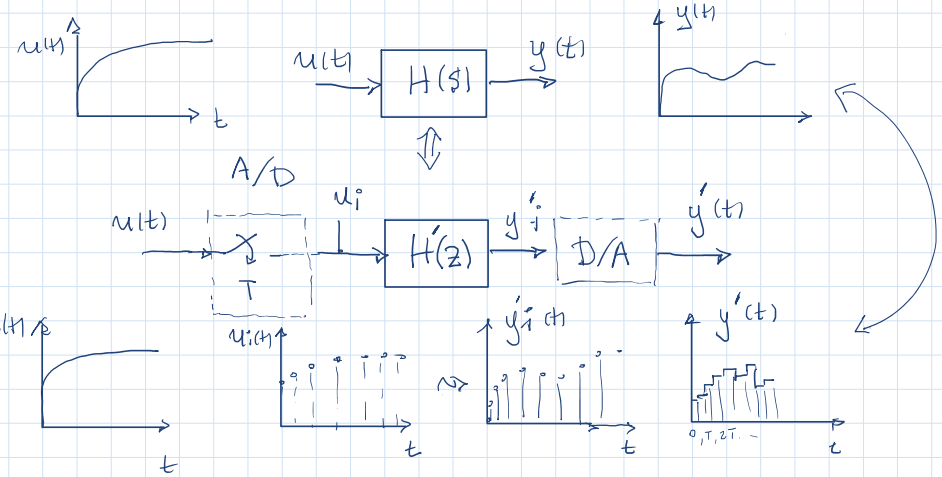
"παράμετρος" ≡ συχν. δειγματοληψίας

- "bandwidth" του συστήματος καθορίζει την



"διάκριτου χρόνου"

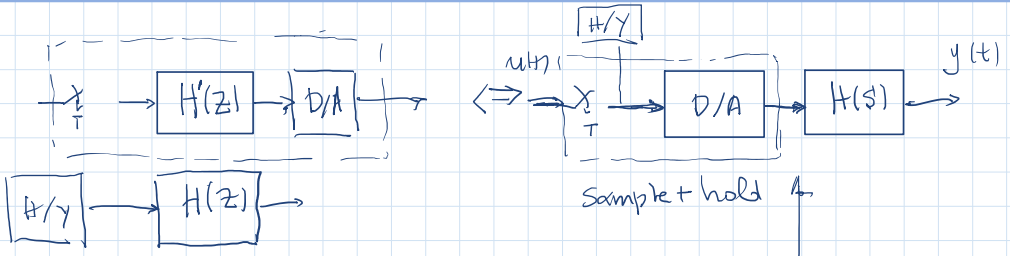
- Πρόβλημα της Διακριτοποίησης Συνεχούς Χρόνου Συστήματος



"Ισοδύναμη σχέση I/O"

$(T \to 0 \Rightarrow y'(t) \approx y(t))$

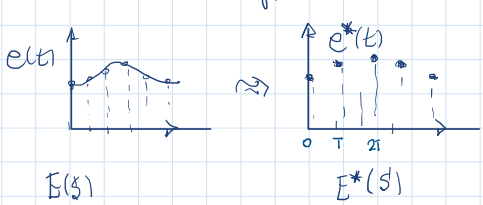
(Διαίσθηση): Πως επιλέγω το $H'(z)$ και D/A έτσι ώστε $y'(t) \approx y(t)$



"δύσκολα διγραμματικών δεδομένων" ανοικτού βρόχου

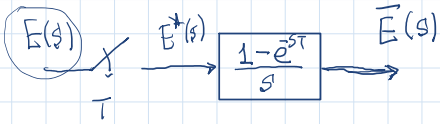
? Σ.Μ. (παλμική) ≈ H(z)

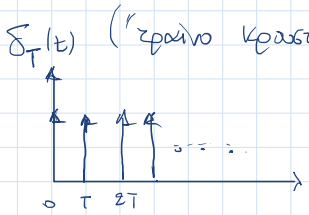
- Ιδανικός Δειγματολήπτης



(* → δειγματοληψία)

$\bar{e}^*(t) = \mathcal{L}^{-1}\{E^*(s)\}$





$\delta_T(t) = \delta(t) + \delta(t-T) + \cdots$

$e^*(t) = e(0)\delta(t) + e(T)\delta(t-T) + e(2T)\delta(t-2T) + \cdots$

$\left(t = nT \Rightarrow e(nT) = e(nT^+) \right)$

ML ⇒ $\boxed{E^*(s) = \sum_{n=0}^{\infty} e(nT)\, e^{-nTs}}$ ⇝ "άρρητη" συνάρτηση ≠ πηλίκο πολυωνύμων

⇒ ~~Σ.Μ.~~ "Ιδαν. δειγματολήπτη"

Παράδειγμα: $e(t) = u(t)$ (μοναδ. βηματική)

$E^*(s) = \sum_{n=0}^{\infty} e(nT)\, e^{-nTs} = e(0) + e(T)e^{-Ts} + e(2T)e^{-2Ts} + \cdots$

$= 1 + 1\cdot e^{-Ts} + 1\cdot e^{-2Ts} + \cdots$

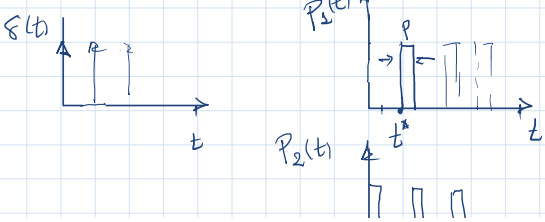
για $|e^{-sT}| < 1$ ⇒ συγκλίνει ⇒ $E^*(s) = \dfrac{1}{1-e^{-sT}}$

Για να βρω Ρητή συνάρτηση:

$E(z) \equiv E^*(s)\Big|_{z=e^{sT}}$

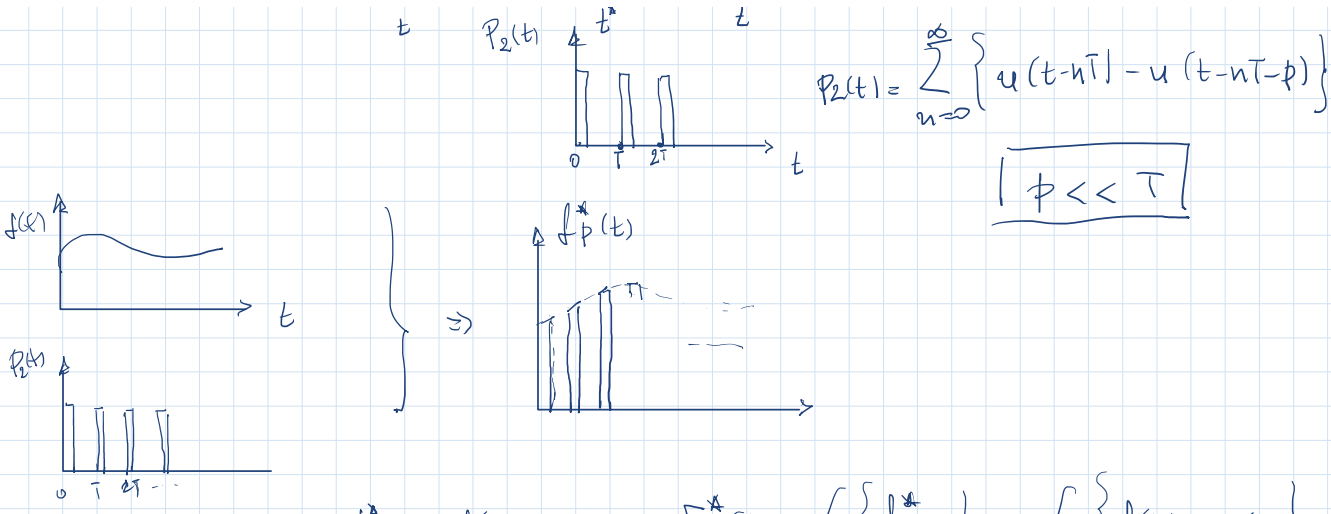
$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs} \Rightarrow \boxed{E(z) = \sum_{n=0}^{\infty} e(nT)\, z^{-n}}$ "Ρητή" συνάρτηση

- Πρακτική Δειγματοληψία



$p_1(t) = u(t-t^*) - u(t-t^*-p)$

$p_2(t) = \sum_{n=0}^{\infty} \{u(t-nT) - u(t-nT-p)\}$



$$f_p^*(t) = f(t)\, p_2(t) \Rightarrow F_p^*(s) = \mathcal{L}\{ f_p^*(t) \} = \mathcal{L}\{ f(t) \cdot p_2(t) \}$$

$$\boxed{F_p^*(s) = \frac{1 - e^{-ps}}{s} F^*(s)}$$

– ΥΠΟΛΟΓΙΣΜΟΣ ΣΗΜΑΤΩΝ ΠΟΥ ΕΧΟΥΝ ΥΠΟΣΤΕΙ ΔΕΙΓΜΑΤΟΛΗΨΙΑ

$$x^*(t) \overset{\text{op}}{=} x(t)\, \delta_T(t) \Rightarrow X^*(s) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nTs}$$

→ 2 τύποι υπολογισμού

$$X^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(s + j n \omega_s) + \frac{x(0)}{2} \qquad \text{(σύσκεψα)}$$

– (Μέθοδος Ολοκληρωτικών Υπολοίπων)

$$X^*(s) = \sum_{1}^{m} \left[\text{Res}\ X(\lambda) \frac{1}{1 - e^{-T(s-\lambda)}} \right]_{\substack{\text{πόλους} \\ \text{των } X(s)}}$$

- Residues k_i : (απλοί πόλοι)

$$k_i = \lim_{\lambda \to \lambda_i} \left[(\lambda - \lambda_i) \frac{X(\lambda)}{1 - e^{-T(s-\lambda)}} \right]$$

- Residues k_i : (πολλ/πλότητας m_i)

$$k_i = \frac{1}{(m-1)!} \lim_{\lambda \to \lambda_i} \frac{d^{m-1}}{d\lambda^{m-1}} \left[(\lambda - \lambda_i) \frac{X(\lambda)}{1 - e^{-T(s-\lambda)}} \right]$$

Παράδειγμα: $E(s) = \dfrac{1}{(s+1)(s+2)}$,

$$E^*(s) = \sum \left[\text{Res}\ E(\lambda) \frac{1}{1 - e^{-T(s-\lambda)}} \right]_{\substack{\text{πόλους} \\ E(\lambda)}} = \frac{1}{(\lambda+2)(1 - e^{-T(s-\lambda)})}\Bigg|_{\lambda=-1} + \frac{1}{(\lambda+1)(1 - e^{-T(s-\lambda)})}\Bigg|_{\lambda=-2}$$

$$= \frac{1}{1 - e^{-T(s+1)}} - \frac{1}{1 - e^{-T(s+2)}}$$

$$= \frac{1}{1-e^{-T(s+1)}} - \frac{1}{1-e^{-T(s+2)}}$$

- $e(t) = \sin\beta t$
- $e(t) = 1 - e^{-t}$

- **Ιδιότητες των ειμάτων $X^*(s)$**

$X^*(s)$ = περιοδικό άπειρο, με περίοδο $j\omega_s$ $\left(\omega_s = \frac{2\pi}{T}\right)$

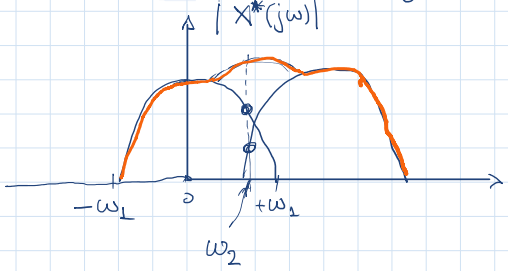
Απόδ: $E^*(s + jm\omega_s) = \sum_{n=0}^{\infty} e(nT) e^{-nT(s + jm\omega_s)}$ $\left(e^{jm\omega_s T} = e^{-jm2\pi} = 1\right)$

$$= \sum_{n=0}^{\infty} e(nT) e^{-nTs} \equiv E^*(s)$$

□ Αν η $E(s)$ έχει $s = s_1 \Rightarrow E^*(s)$ έχει πόλους $s = s_1 + jm\omega_s$

$m = 0, \pm 1, \pm 2, \dots$

Εξομάδωση ("Aliasing")



- $\omega = \omega_2 \Rightarrow |X^*(j\omega_2)| + |X^*(j(\omega_s - \omega_2)|$

$|n\omega_s \pm \omega_2$

π.χ. $x(kT) = \sin(\omega_2 kT + \theta)$

$y(kT) = \sin[(\omega_2 + n\omega_s)kT + \theta]$

$= \sin(\omega_2 kT) + 2\pi k.n + \theta)$

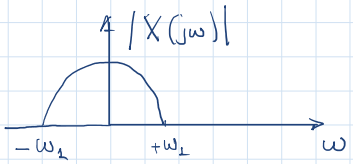
$= \sin(\omega_2 kT + \theta)$

$\Rightarrow x(kT) \equiv y(kT)$

ΘΕΩΡΗΜΑ SHANNON

Έστω $e(t) \sim f_0$ (Hz), $f_s \equiv \frac{1}{2f_0}$

ΑΝΑΚΑΤΑΣΚΕΥΗ ΔΕΔΟΜΕΝΩΝ



$x(t) \Rightarrow$ "φάσμα $x(t)$"

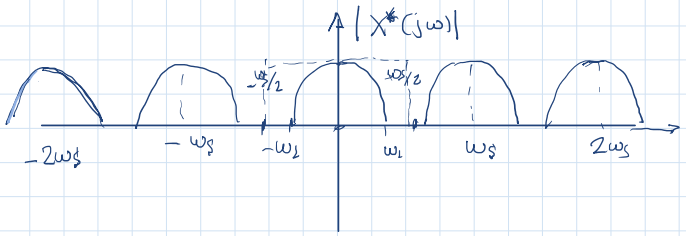
$x(t) \rightarrow x^*(t)$ $\quad X^*(s) = \mathcal{L}[x^*(t)]$ $\quad |X^*(j\omega)| = ?$

$\left(\omega_s = \frac{2\pi}{\dots}\right)$

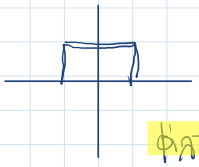
$$X^*(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j\omega + jn\omega_s) \qquad \left(\omega_s = \frac{2\pi}{T}\right)$$

$$= \cdots + \frac{1}{T}X(j(\omega-\omega_s)) + \frac{1}{T}X(j\omega) + \frac{1}{T}X(j(\omega+\omega_s)) + \cdots$$

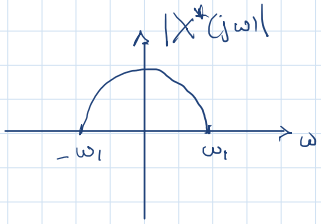
□ $\omega_s > 2\omega_1$



$\Rightarrow$



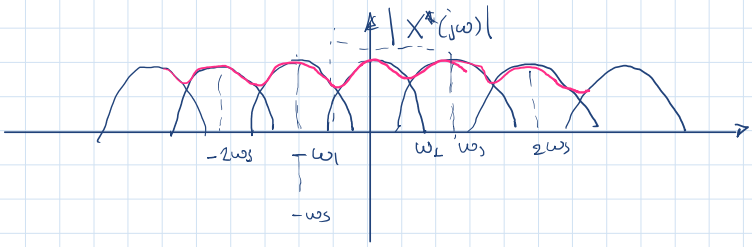
φίλτρο



"ανακατασκευαστής"

D/A

□ $\omega_s < 2\omega_1$



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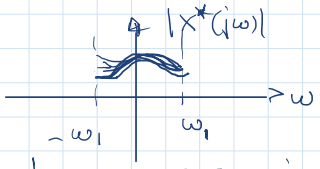
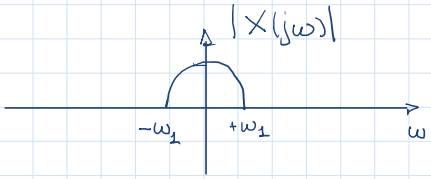


- "Εξομάλυνση" → Συχνότητα $> \frac{1}{2}\omega_s$ στα σήματα $\left(0, \frac{\omega_s}{2}\right)$
- D/A ≡ ανακατασκευαστής ≡ φίλτρο

□ Ανακατασκευαστής



Έστω $x(t)$:

$x(t) \xrightarrow[T]{} x^*(t) \longrightarrow$

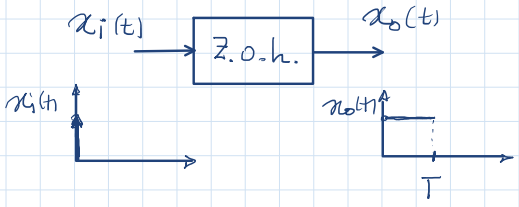
Θ/: $x(t) \simeq x(nT) + \frac{\dot{x}(nT)}{1!}(t-nT) + \frac{\ddot{x}(nT)}{2!}(t-nT)^2 + \cdots$ (σ. Taylor)

"προσέγγιση": $\dot{x}(nT) \simeq \frac{1}{T}\left[x(nT) - x[n-1]T\right]$

$$\ddot{x}(nT) \simeq \frac{1}{T}\left[\dot{x}(nT) - \dot{x}[(n-1)T]\right] = \cdots = \frac{1}{T^2}\left[x(nT) - 2x[(n-1)T] + x[(n-2)T]\right]$$

ΜΗΔΕΝΙΚΟΣ ΑΝΑΚΑΤΑΣΚΕΥΑΣΤΗΣ

$$x_\eta(t) \approx x(nT) \quad , \quad nT \le t \le (n+1)T$$



$$x_o(t) = u(t) - u(t-T)$$

$$\overset{ML}{\Rightarrow} \quad X_o(s) = \frac{1}{s} - \frac{e^{-sT}}{s}$$

$$X_i(s) = 1$$

$$\Rightarrow \boxed{\frac{X_o(s)}{X_i(s)} = \frac{1-e^{-sT}}{s}}$$

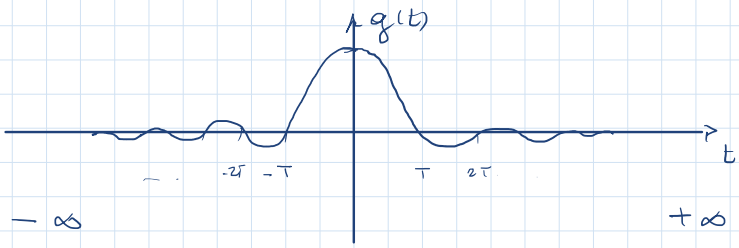
- Ανακατασκευαστής 1ης τάξης : $x_\eta(t) \cong x(nT) + \frac{x'(nT)}{1!}(t-nT)$ ⇝

□ Ιδανικό φίλτρο Χαμηλοδιαβατό ⇝ "Μή Υλοποιήσιμο"

$$G(j\omega) = \begin{cases} 1 & -\frac{1}{2}\omega_s \le \omega \le \frac{1}{2}\omega_s \\ 0 & \text{άλλου} \end{cases}$$

$$\overset{MF^{-1}}{\Rightarrow} \quad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)\, e^{j\omega t}\, d\omega = \cdots = \frac{1}{\pi t}\sin\frac{\omega_s t}{2}$$

$$\Rightarrow \quad g(t) = \frac{1}{T}\,\frac{\sin\left(\frac{\omega_s t}{2}\right)}{\frac{\omega_s t}{2}}$$



απόκριση $-\infty \to +\infty$

$\exists$ -//- $t < 0$ άρα

πρίν εφαρμοσθεί σίσοδος υπάρχει απόκριση

Άσκηση: $X(s) = \dfrac{10s+4}{s^3+6s^2+5s}$ ⇝ $X^*(s) = ?$

α) $x(t) = \mathcal{L}^{-1}\left\{\dfrac{10s+4}{s^3+6s^2+5s}\right\} = \mathcal{L}^{-1}\left\{\dfrac{10s+4}{s(s+1)(s+5)}\right\}$

$$X(s) = \frac{10s+4}{s(s+1)(s+5)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+1} + \frac{\alpha_3}{s+5}$$

$$\alpha_1 = \lim_{s\to 0} X(s)\cdot s = \lim_{s\to 0}\frac{10s+4}{\cancel{s}(s+1)(s+5)}\cdot \cancel{s} = \frac{4}{5}$$

$$\alpha_2 = \qquad = \frac{3}{2}$$

$\alpha_2 = \qquad\qquad\qquad\qquad = \frac{3}{2}$

$\alpha_3 = \qquad\qquad\qquad\qquad = -\frac{23}{10}$

Άρα, $x(t) = \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{23}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$

$$= \frac{4}{5} + \frac{3}{2}e^{-t} - \frac{23}{10}e^{-5t}$$

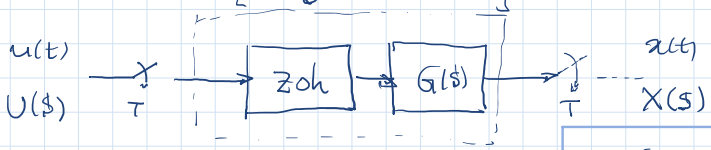
$$x^*(t) = x(t)\,\delta_T(t) = \sum_{k=0}^{\infty}\left(\frac{4}{5} + \frac{3}{2}e^{-t} - \frac{23}{10}e^{-5t}\right)\delta_T(t)$$

$$\Rightarrow X^*(s) = \sum_{k=0}^{\infty}\left(\frac{4}{5} + \frac{3}{2}e^{-kT} - \frac{23}{10}e^{-5kT}\right)\left(e^{sT}\right)^k \quad \ldots$$

$$X(z) \overset{op.}{=} X^*(s)\Big|_{z=e^{sT}} = \ldots\ldots$$

Άσκηση: Υπολογισμός Μ.Ζ. συναρτήσεων με τόν ὄρο $\frac{1-e^{-sT}}{s}$

$$X(s) = \left[\frac{1-e^{-sT}}{s}\,G(s)\right] \quad \rightsquigarrow \quad X(z) = ?$$



$$G(z) = \mathcal{Z}\left\{G_{ho}(s)\,G(s)\right\} = \boxed{\mathcal{Z}\left\{\frac{1-e^{-sT}}{s}G(s)\right\} = \left(1-z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\}}$$

Απόδειξη:

Θέτω $\quad G_1(s) \triangleq \frac{G(s)}{s}$

$$G'(s) = \left(1-e^{-sT}\right)G_1(s) \equiv G_1(s) - e^{-sT}G_1(s)$$

Θέτω $\quad X_1(s) = e^{-sT}G_1(s)$

$$\overset{ML^{-1}}{\Rightarrow} \quad x_1(t) = \int_0^t g_0(t-\tau)\,g_1(\tau)\,d\tau \qquad \left(g_0(t) = \mathcal{L}^{-1}\left\{e^{-sT}\right\} = \delta(t-T)\right)$$

$$g_1(t) = \mathcal{L}^{-1}\left\{G_1(s)\right\}$$

Άρα, $x_1(t) = \int_0^T \delta(t-T)\,g_1(\tau)\,d\tau = g_1(t-T)$

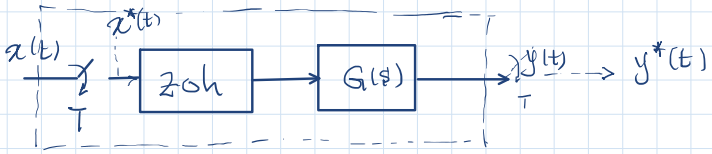
$$\Rightarrow \mathcal{Z}\left\{x_1(t)\right\} = \mathcal{Z}\left\{g_1(t-T)\right\} = z^{-1}G_1(z)$$

Τότε, $\quad G'(z) = \mathcal{Z}\left\{G_1(s) - e^{-sT}G_1(s)\right\} = \mathcal{Z}\left\{g_1(t)\right\} - \mathcal{Z}\left\{x_1(t)\right\}$

$= G_1(z) - z^{-1} G_1(z)$

Άρα, $\boxed{G'(z) = (1 - z^{-1}) G_1(z)}$

Άσκηση: $G(s) = \dfrac{1}{s(s+1)}$



$G(z) = ??$

α) Μέσω Z (τύπους)

$$G(z) = (1 - z^{-1}) \mathcal{Z}\left\{ \frac{1}{s^2(s+1)} \right\} = (1 - z^{-1}) \mathcal{Z}\left\{ \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right\}$$

$$\Rightarrow \boxed{G(z) = \frac{(T - 1 - e^{-T}) z^{-1} + (1 - e^{-T} - Te^{-T}) z^{-2}}{(1 - z^{-1})(1 - e^{-T} z^{-1})}}$$

β) Άμεσα Κλάσματα

$$G(s) = (1 - e^{-sT}) \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right)$$

$\xrightarrow{ML^{-1}}$ $g(t) = (t - 1 + e^{-t}) u(t)$ (

$$G(z) = \sum_{k=0}^{\infty} g(kT) z^{-k} = \dots$$

γ) Ολοκληρωτικά Υπόλοιπα:

$$G(z) = (1 - z^{-1}) \mathcal{Z}\left\{ \frac{1}{s^2(s+1)} \right\}$$

Θέτω $X(s) = \dfrac{1}{s^2(s+1)} \rightarrow X(z) = ?$

↳ 1 διπλός πόλος $s = 0$

1 απλός πόλος $s = -1$

$$X(z) = \sum \text{Res} \left. \frac{X(s) z}{z - e^{sT}} \right|_{\substack{\text{πόλοι} \\ X(s)}} = \frac{1}{(2-1)!} \lim_{s \to 0} \frac{d}{ds} \left[s^2 \frac{1}{s^2(s+1)} \frac{z}{z - e^{sT}} \right] +$$

$$+ \lim_{s \to -1} \left[(s+1) \frac{1}{s^2(s+1)} \frac{z}{z - e^{sT}} \right]$$

$= \dots$