ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΑΤΡΩΝ

ΤΜΗΜΑ ΟΙΚΟΝΟΜΙΚΩΝ ΕΠΙΣΤΗΜΩΝ

ΜΑΘΗΜΑΤΙΚΑ ΓΙΑ ΟΙΚΟΝΟΜΟΛΟΓΟΥΣ ΙΙ

ΕΡΓΑΣΤΗΡΙΟ-ΦΡΟΝΤΙΣΤΗΡΙΟ 3^{Ο}

<u>**ΜΕΡΙΚΗ ΠΑΡΑΓΩΓΟΣ-ΔΙΑΝΥΣΜΑ ΚΛΙΣΗΣ-ΙΑΚΩΒΙΑΝΗ ΜΗΤΡΑ**</u>

Άσκηση 1

Να βρεθούν οι μερικές παράγωγοι των παρακάτω συναρτήσεων:

i. $f(x,y) = 2xy^2$

$$\frac{\partial f(x,y)}{\partial x} = 2y^2, \qquad \frac{\partial f(x,y)}{\partial y} = 2x\,2y = 4xy$$

ii. $f(x,y) = x^2 + xy + y^2$

$$\frac{\partial f(x,y)}{\partial x} = 2x + y, \frac{\partial f(x,y)}{\partial y} = x + 2y$$

iii. $f(x,z) = x^2 + 3z^3$

$$\frac{\partial f(x,z)}{\partial x} = 2x, \frac{\partial f(x,z)}{\partial z} = 9z^2$$

iv. $f(x,y,z) = xyz$

$$\frac{\partial f(x,y,z)}{\partial x} = yz, \frac{\partial f(x,y,z)}{\partial y} = xz, \frac{\partial f(x,y,z)}{\partial z} = xy$$

v. $f(x,y,z) = x^2 + ye^z$

$$\frac{\partial f(x,y,z)}{\partial x} = 2x, \frac{\partial f(x,y,z)}{\partial y} = e^z, \frac{\partial f(x,y,z)}{\partial z} = ye^z$$

vi. $f(x,y) = e^x \sin xy$

$$\frac{\partial f(x,y)}{\partial x} = e^x \sin(xy) + e^x y \cos(xy), \frac{\partial f(x,y)}{\partial y} = e^x x \cos(xy)$$

vii. $f(x,y) = \frac{xy(x^2+y^2)}{x^2+y^2+1}$

$$\frac{\partial f\left(\frac{xy(x^2+y^2)}{x^2+y^2+1}\right)}{\partial x} = \frac{\left(xy\left(x^2+y^2\right)\right)'(x^2+y^2+1)-(x^2+y^2+1)'\left(xy\left(x^2+y^2\right)\right)}{(x^2+y^2+1)^2} =$$

$$\frac{(3yx^2+y^3)(x^2+y^2+1)-2x\left(xy\left(x^2+y^2\right)\right)}{(x^2+y^2+1)^2} = \frac{3yx^4+x^2y^3+3y^3x^2+y^5+3yx^2+y^3-2x^4y-2x^2y^3}{(x^2+y^2+1)^2} =$$

$$\frac{yx^4+2x^2y^3+y^5+3yx^2+y^3}{(x^2+y^2+1)^2} = \frac{y(x^4+2x^2y^2+y^4+3x^2+y^2)}{(x^2+y^2+1)^2}$$

$$\frac{\partial f\left(\frac{xy(x^2+y^2)}{x^2+y^2+1}\right)}{\partial y} = \frac{\left(xy\left(x^2+y^2\right)\right)'(x^2+y^2+1)-(x^2+y^2+1)'\left(xy\left(x^2+y^2\right)\right)}{(x^2+y^2+1)^2} =$$

$$\frac{(3xy^2+x^3)(x^2+y^2+1)-2y(yx^3+xy^3)}{(x^2+y^2+1)^2} = \frac{x^5+3x^3y^2+x^3y^2+3xy^4+x^3+3xy^2-2x^3y^2-2xy^4}{(x^2+y^2+1)^2}$$

$$\frac{x^5+2x^3y^2+x^3+yx^4+3xy^2}{(x^2+y^2+1)^2} = \frac{x(x^4+2x^2y^2+y^4+3y^2+x^2)}{(x^2+y^2+1)^2}$$

viii. $f(x,y) = \sin(xy) + 3xy$

$$\frac{\partial f(x,y)}{\partial x} = (\sin(xy) + 3xy)' = y\cos(xy) + 3y$$

$$\frac{\partial f(x,y)}{\partial y} = (\sin(xy) + 3xy)' = x\cos(xy) + 3x$$

ix. $f(x,y,z) = 3x^2zy + 2xy^2z^2 + 3z + 2y + x + 5$

$$\frac{\partial f(x,y,z)}{\partial x} = 6xzy + 2y^2z^2 + 0 + 0 + 1 + 0$$

$$\frac{\partial f(x,y,z)}{\partial y} = 3x^2z + 4xyz^2 + 0 + 2 + 0 + 0$$

$$\frac{\partial f(x,y,z)}{\partial z} = 3x^2y + 4xy^2z + 3 + 0 + 0 + 0$$

x. $Q(K,L) = AK^aL^{1-a}\ a > 0, 0 < a < 1$

$$MP_K = \frac{\partial Q(K,L)}{\partial K} = aAK^{a-1}L^{1-a}$$

$$MP_L = \frac{\partial Q(K,L)}{\partial L} = (1-a)AK^aL^{-a}$$

$$MRTS_{KL} = -\frac{MP_L}{MP_K} = -\frac{(1-a)AK^aL^{-a}}{aAK^{a-1}L^{1-a}} = -\frac{(1-a)K}{aL}$$

xi. $Q(K,L) = AK^aL^b \ a > 0, 0 < a, b < 1$

$$MP_K = \frac{\partial Q(K,L)}{\partial K} = aAK^{a-1}L^b$$

$$MP_L = \frac{\partial Q(K,L)}{\partial L} = bAK^aL^{b-1}$$

$$MRTS_{KL} = -\frac{MP_L}{MP_K} = -\frac{bAK^aL^{b-1}}{aAK^{a-1}L^b} = -\frac{bK}{aL}$$

xii. $U(x,y) = x^a y^{1-a}\ ,\ 0 < a < 1$

$$MU_x = \frac{\partial U(x,y)}{\partial x} = ax^{a-1}y^{1-a}$$

$$MU_y = \frac{\partial U(x,y)}{\partial L} = x^a(1-a)y^{-a}$$

$$MRS_{yx} = -\frac{MU_x}{MU_y} = -\frac{ax^{a-1}y^{1-a}}{x^a(1-a)y^{-a}} = -\frac{ay}{(1-a)x}$$

xiii. $U(x,y) = aln(x) + bln(y) \quad a, b > 0$

$$MU_x = \frac{\partial U(x,y)}{\partial x} = \frac{a}{x}$$

$$MU_y = \frac{\partial U(x,y)}{\partial L} = \frac{b}{y}$$

$$MRS_{yx} = -\frac{MU_x}{MU_y} = -\frac{\frac{a}{x}}{\frac{b}{y}} = -\frac{ay}{bx}$$

Άσκηση 2

Έστω οι παρακάτω συναρτήσεις:

i. $U(x,y) = y^a x^b$
ii. $U(x,y) = aln(y-c) + b(x-d)$
iii. $Q(K,L,E) = K^aL^bE^c$

Να βρεθούν τα διανύσματα κλίσης των παραπάνω συναρτήσεων

i. $U(y,x) = y^a x^b, \nabla U = \begin{pmatrix} \frac{\partial U}{\partial y} \\ \frac{\partial U}{\partial x} \end{pmatrix} = \begin{pmatrix} a y^{a-1} x^b \\ b x^{b-1} y^a \end{pmatrix}$

ii. $U(y,x) = a ln(y-c) + b(x-d), \nabla U = \begin{pmatrix} \frac{a}{y-c} \\ b \end{pmatrix}$

iii. $Q(K,L,E) = K^a L^b E^c, \nabla U = \begin{pmatrix} aK^{a-1}L^b E^c \\ \beta K^a L^{b-1} E^c \\ \gamma K^a L^b E^{c-1} \end{pmatrix}$

Άσκηση 3

Να εξεταστούν οι παρακάτω ομάδες συναρτήσεων, ως προς την γραμμική ή μη γραμμική τους εξάρτηση.

i. $f(x,z) = x + 2z$

ii. $g(x,z) = x^4 + 8x^3 z + 24x^2 z^2 + 32xz^3 + 16z^4$

Για να εξετάσουμε τη συναρτησιακή εξάρτηση των ομάδων συναρτήσεων, θα πρέπει να υπολογίσουμε την τιμή της ιακωβιανής ορίζουσας.

$$J_{x,z} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial z} \end{bmatrix} =>$$

$$=> \begin{bmatrix} 1 & 2 \\ (4x^3 + 24x^2 z + 48xz^2 + 32z^3) & (8x^3 + 48x^2 z + 96xz^2 + 64z^3) \end{bmatrix} =>$$

$$(8x^3 + 48x^2 z + 96xz^2 + 64z^3) - 2(4x^3 + 24x^2 z + 48xz^2 + 32z^3) = 0$$

Άρα υπάρχει συναρτησιακή εξάρτηση.