LECTURE 1- PRODUCTION, TECHNOLOGY AND COST FUNCTIONS (BASIC DEFINITIONS) MEASUREMENT

Konstantinos Kounetas

School of Business Administration
Department of Economics

Master of Science in Applied Economic Analysis

Objectives for the first 3-4 weeks

1. Examine the conceptual framework that underpins productivity and efficiency measurement
2. Introduce two principal methods
   - Data Envelopment Analysis
   - Stochastic Frontier Analysis
   - We leave indices (Laspeyers, Paasche, Tornqvist)

Examine these techniques, relative merits, necessary assumptions and guidelines for their applications

Objectives for the first 3-4 weeks

3. Work with computer programs (we will use these in forthcoming sessions)
   - DEAP, R
   - FRONTIER, STATA
4. Briefly review some case studies and real life applications

Outline for today

- Introduction to Production Functions
- Concepts
  - Assumptions
  - Stages of Production
  - Long and Short run-Isoquants
  - Returns to scale
  - Cost Minimization and Profit maximization
  - Shepard’s and Hotelling’s Lemma
  - Econometric specification

Some Informal Definitions

- Productivity
- Technical Efficiency
- Allocative Efficiency
- Technical Change
- Scale Economies
- Total Factor Productivity
- Production frontier
- Feasible Production Set

Introduction

- Performance measurement
  - Productivity measures
  - Benchmarking performance
    - Mainly using partial productivity measures
    - Cost, revenue and profit ratios
    - Performance of public services and utilities
- Aggregate Level
  - Growth in per capita income
  - Labour and total factor productivity growth
  - Sectoral performance
    - Labour productivity
    - Share in the total economy
- Industry Level
  - Performance of firms and decision making units (DMUs)
  - Market and non-market goods and services
  - Efficiency and productivity
  - Banks, credit unions, manufacturing firms, agricultural farms, schools and universities, hospitals, aged care facilities, etc.
- Need to use appropriate methodology to benchmark performance

Some Basic Issues I

- A fundamental requirement in applying operations research models is the identification of a "utility function" which combines all variables relevant to a decision problem into a single variable which is to be optimized. Underlying the concept of a utility function is the view that it should represent the decision-maker's perceptions of the relative importance of the variables involved rather than being regarded as uniform across all decision-makers or externally imposed.
- The problem, of course, is that the resulting utility functions may bear no relationship to each other and it is therefore difficult to make comparisons from one decision context to another. Indeed, not only may it not be possible to compare two different decision-makers but it may not be possible to compare the utility functions of a single decision-maker from one context to another.
- A traditional way to combine variables in a utility function is to use a cost/effectiveness ratio, called an "efficiency" measure

SOME BASIC ISSUES II

- Efficiency:
  - (i)How much more can we produce with a given level of inputs?
  - (ii) How much input reduction is possible to produce a given level of observed output?
  - (iii) How much more revenue can be generated with a given level of inputs? Similarly how much reduction in input costs be achieved?
- Productivity:
  - We wish to measure the level of output per unit of input and compare it with other firms
  - Partial productivity measures – output per person employed; output per hour worked; output per hectare etc.
  - Total factor productivity measures – Productivity measure which involves all the factors of production
    - More difficult to conceptualise and measure

Simple performance measures

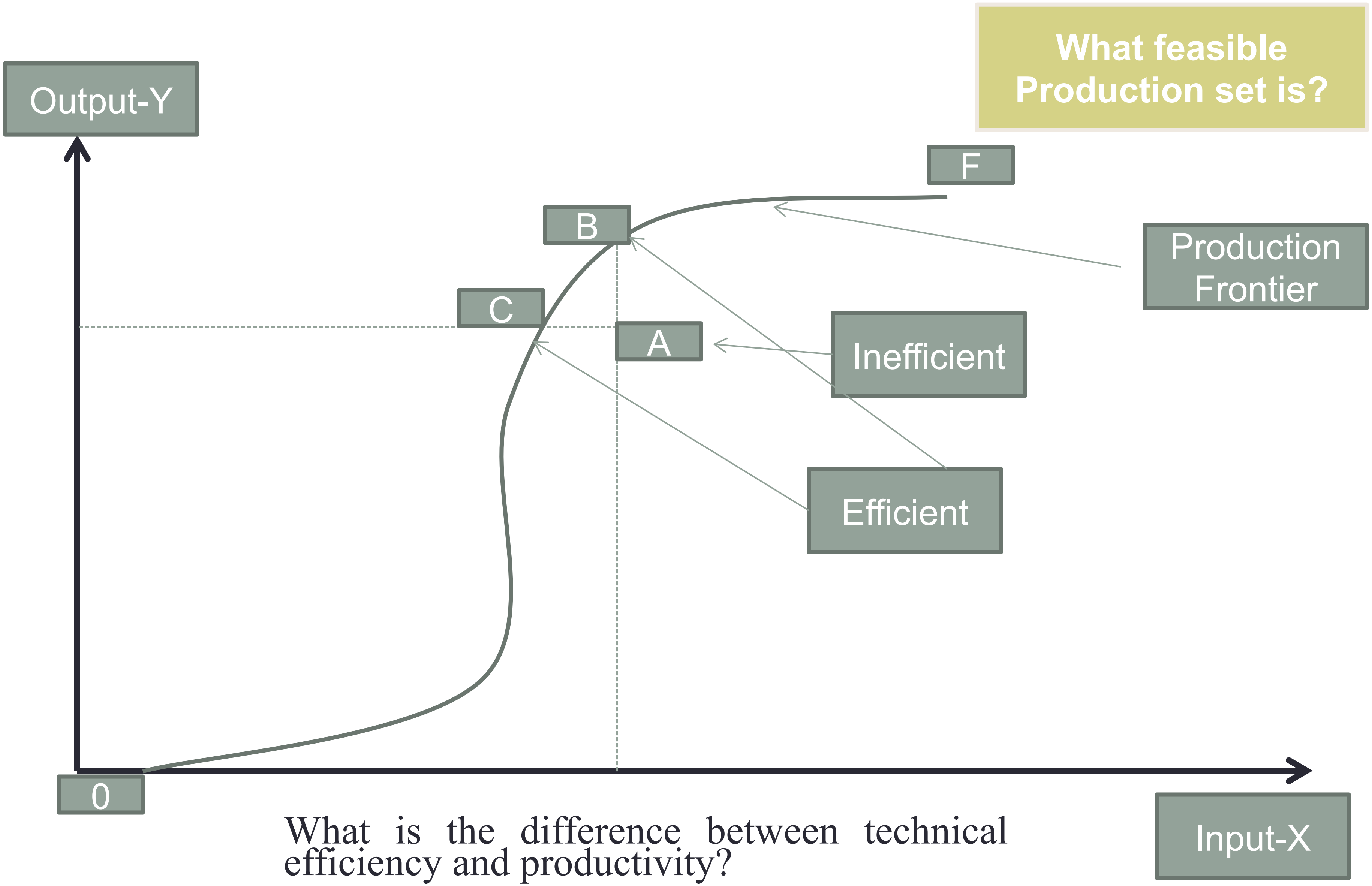
- Can be misleading
- Consider two clothing factories (*A* and *B*)
- *Labour productivity* could be higher in firm *A* – but what about use of capital and energy and materials?
- *Unit costs* could be lower in firm *B* – but what if they are located in different regions and face different input prices?

Terminology?

- The terms *productivity* and *efficiency* relate to similar (but not identical) things
- Productivity = output/input
- Efficiency generally relates to some form benchmark or target
- A simple example – where for firm *B* productivity rises but efficiency falls:

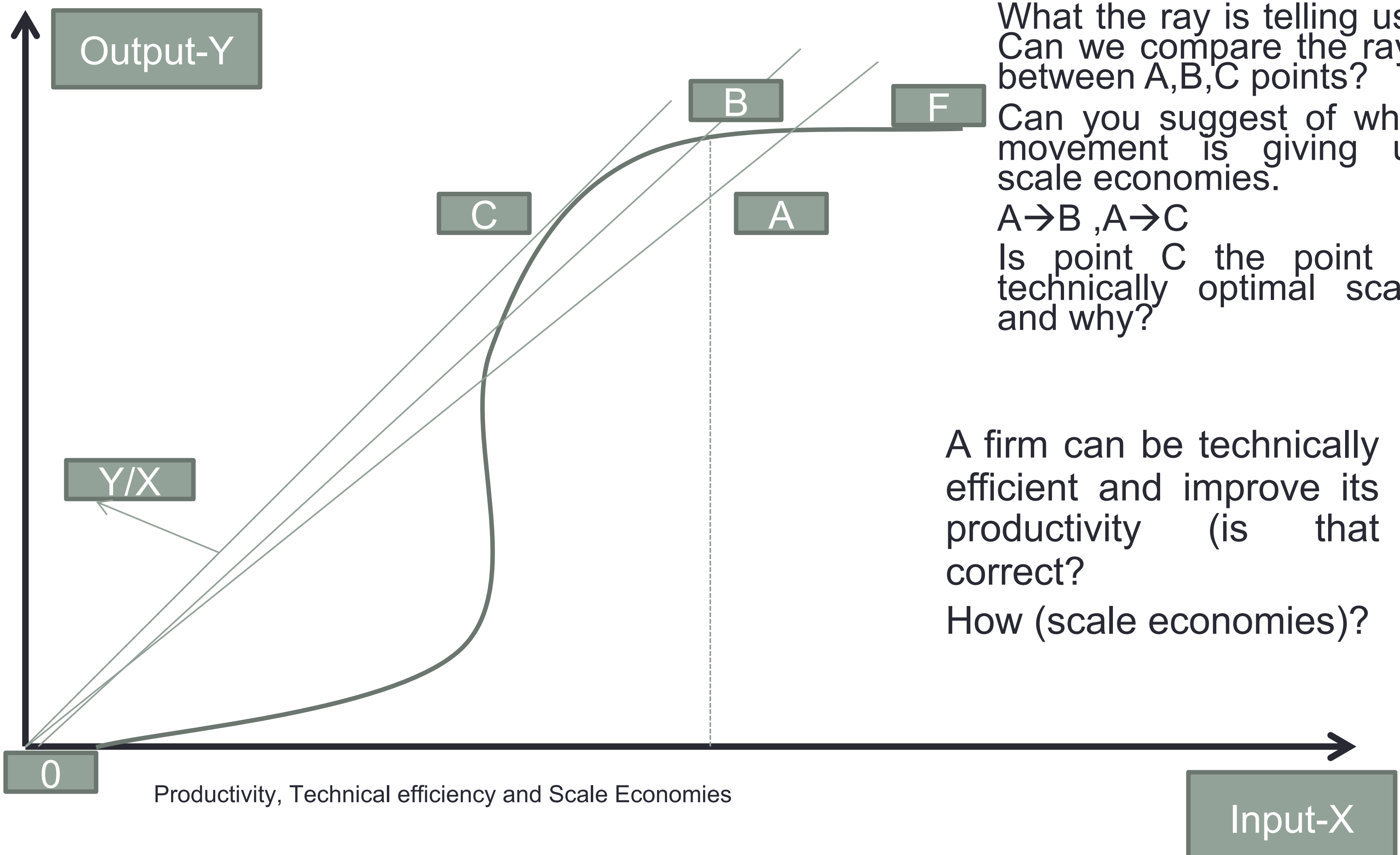
firm	year	input	output	productivity	efficiency
A	1	2	6	3	0.75
B	1	4	16	4	1.00
A	2	2	8	4	0.67
B	2	5	30	6	1.00

Production frontier and technical efficiency



What is the difference between technical efficiency and productivity?

Production frontier, technical efficiency and scale economy



What the ray is telling us? Can we compare the rays between A,B,C points?

Can you suggest of what movement is giving us scale economies.

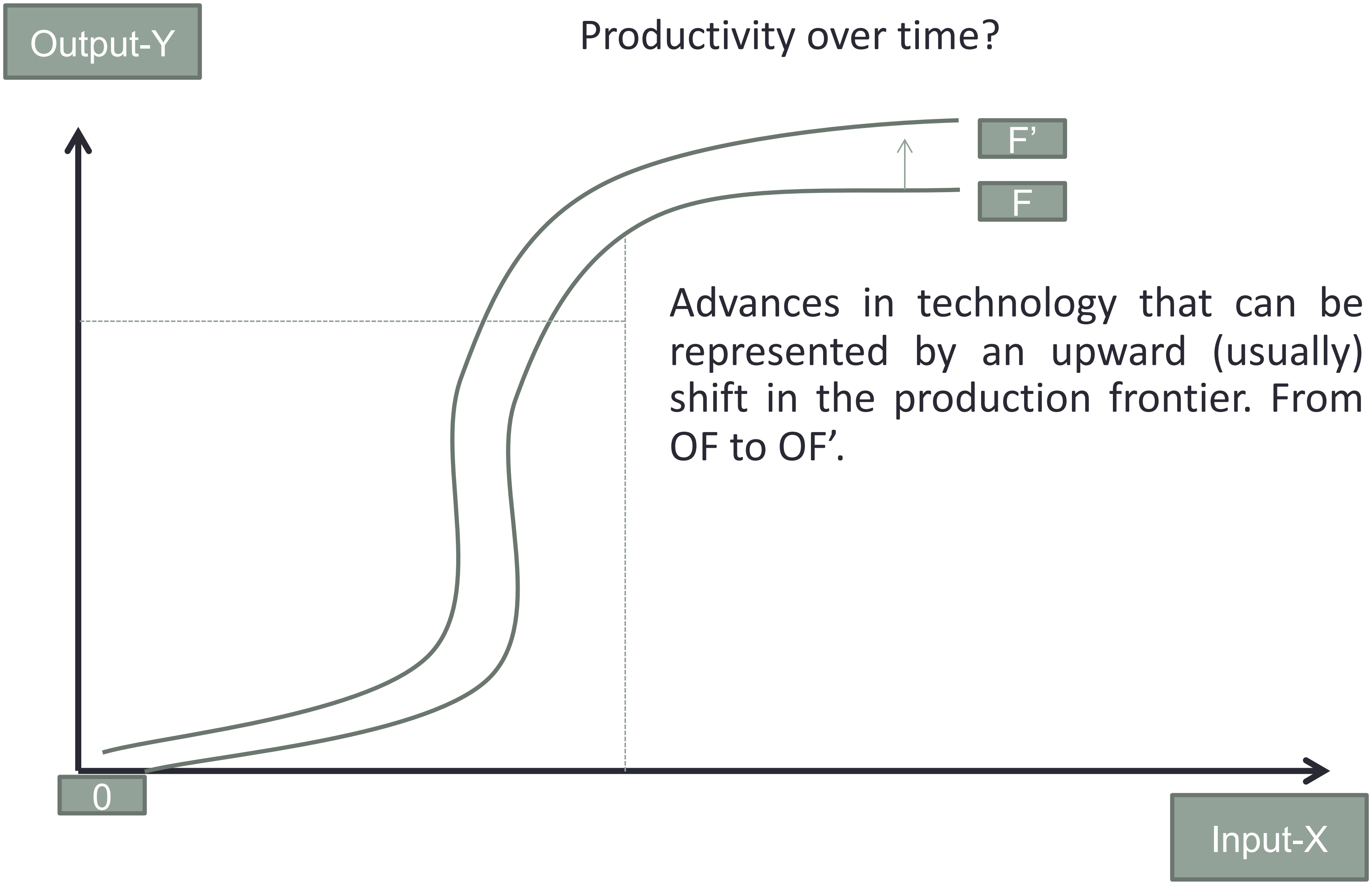
A→B ,A→C

Is point C the point of technically optimal scale and why?

A firm can be technically efficient and improve its productivity (is that correct?

How (scale economies)?

Technical change



Productivity over time?

Advances in technology that can be represented by an upward (usually) shift in the production frontier. From OF to OF'.

Productivity

- One of the most important responsibilities of an operations manager is to achieve productive use of organization's resources. When we refer to productivity we mean **total factor productivity (TFP)**.

- **Productivity** is an index that measures output (goods and services) relative to the input (capital, labor, materials, energy, and other resources) used to produce them.

- It is usually expressed as the ratio of output to input:

$$\text{Productivity} = \frac{Output}{Input}$$

Ways to Increase Productivity

- Increase output by using the same or a lesser amount of (input) resource.
- Reduce amount of (input) resource used while keeping output constant or increasing it.
- Use more resource as long as output increases at a greater rate.
- Decrease output as long as resource use decreases at a greater rate.
- Production is concerned with the activity of producing goods and services.
- Productivity is concerned with the efficiency and effectiveness with which these goods and services are produced.

Efficiency and Effectiveness for productivity improvement.

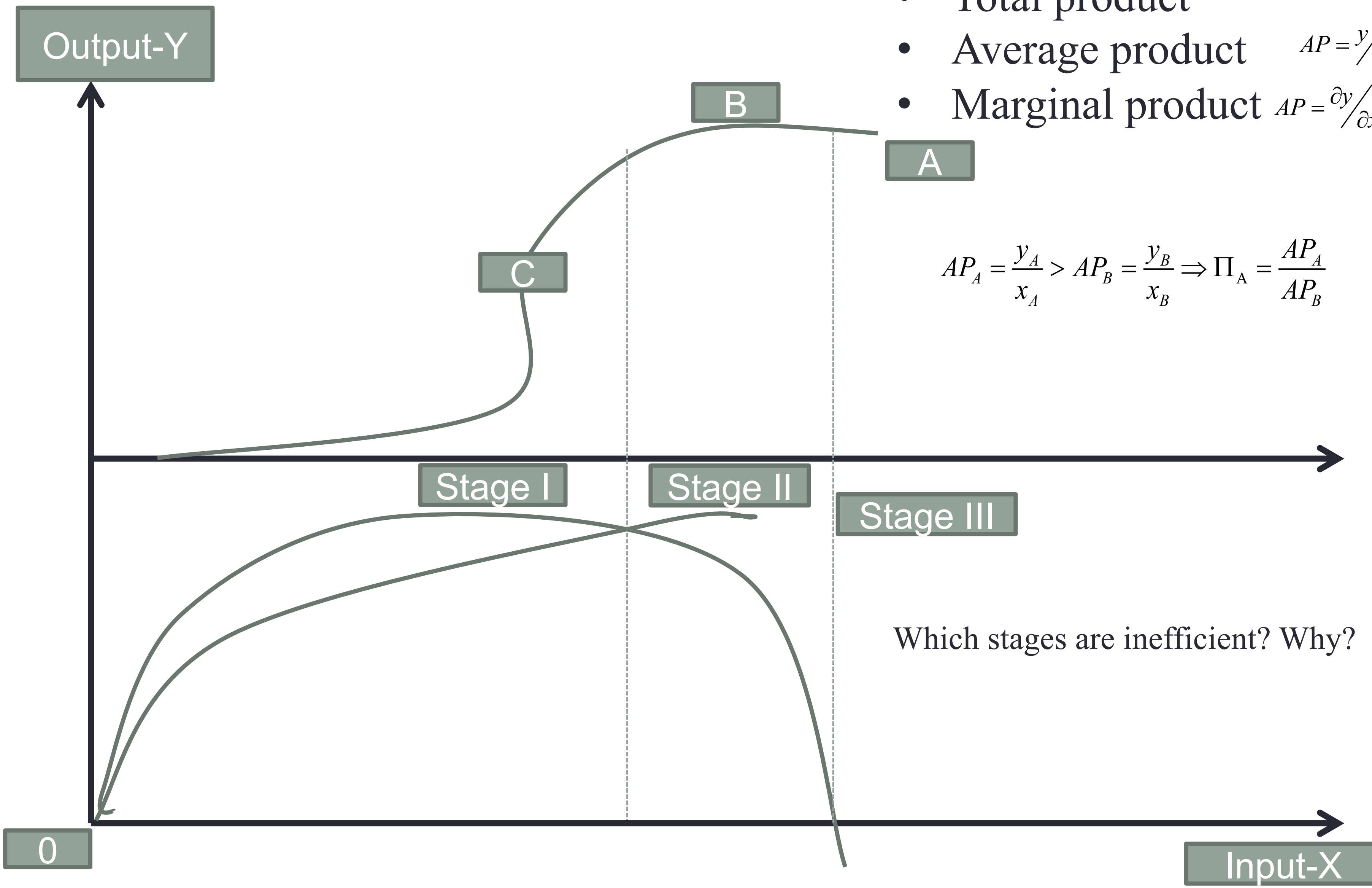
- Efficiency is a necessary but not a satisfactory condition for productivity. In fact, both effectiveness and efficiency are necessary in order to be productive.
- Efficiency is the ratio of actual output generated to the expected (or standard) output prescribed.
- Effectiveness, on the other hand, is the degree to which the relevant goals or objectives are achieved.
- Effectiveness involves first determining the relevant (right) goals or objectives and then achieving them.
    - If, for example, nine out of ten relevant goals are achieved, the effectiveness is 90%. One can be very efficient and still not be productive.

Introduction to Production Economics

- Production function (Varian 1992; Fare and Primont, 1995) describes the technical relationship between the inputs and the outputs of a production process. Gives the maximum output(-s) attainable from a given vector of inputs.
- Usually the mathematical formulation is the following $y \equiv Q = f(x_1, x_2, \ldots, x_n)$
- For simplicity reasons we assume $y \equiv Q = f(x_1, x_2)$ and considering that, for example, x_2 is fixed $y \equiv Q = f(x_1 | x_2 = x_{20})$
- The production process is monoperiodic
- Input-Outputs are homogeneous
- Twice differentiable
- Production function, input and output prices are known
- No budget constraint exists
- Maximizing profit is the goal of an entity (firm, industry, country e.t.c)

Product Curves-3 stages of production

- Total product
- Average product $AP = {}^{y}\!/_{x}$
- Marginal product $AP = {}^{\partial y}\!/_{\partial x}$

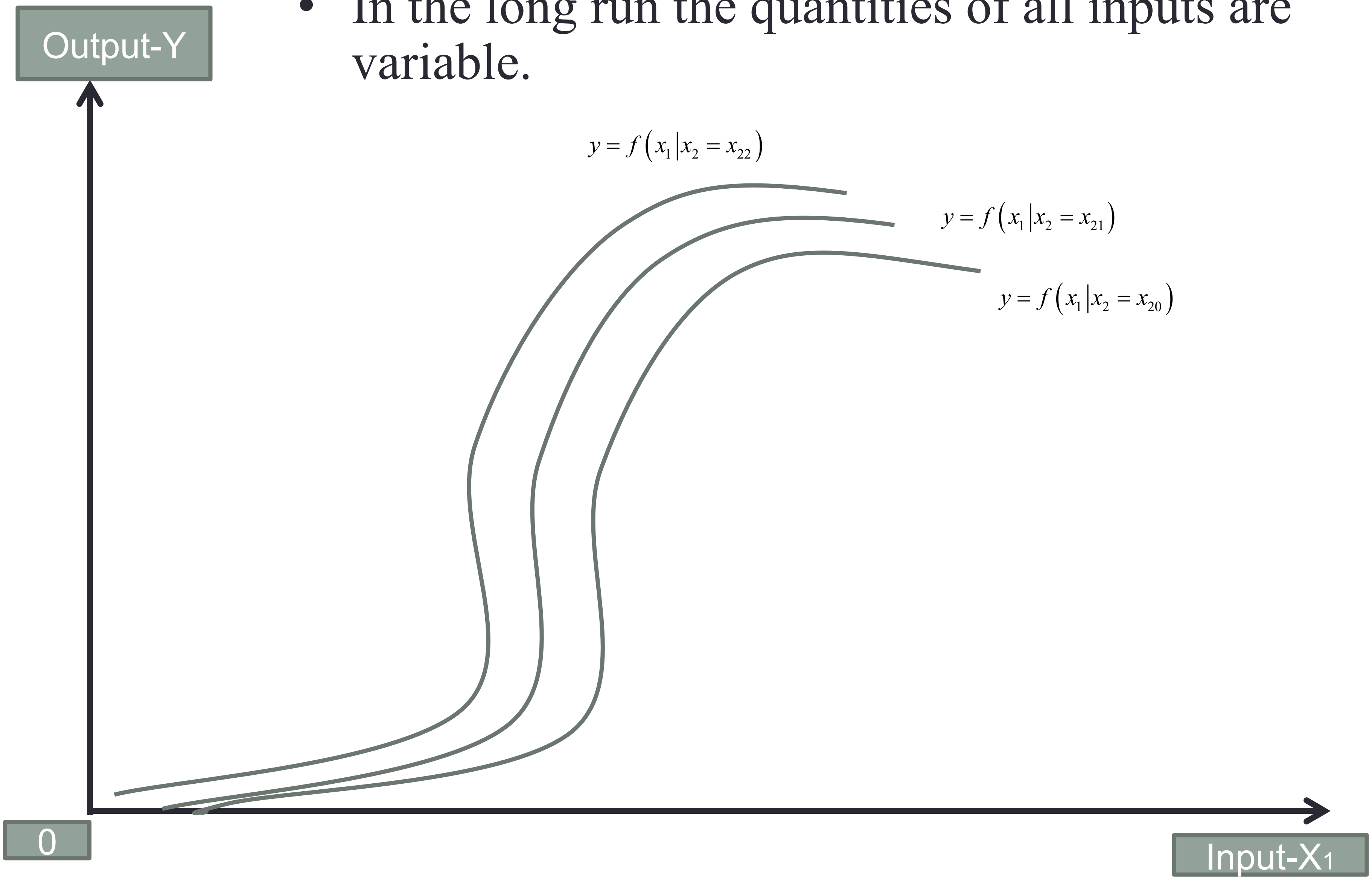


$$AP_A = \frac{y_A}{x_A} > AP_B = \frac{y_B}{x_B} \Rightarrow \Pi_A = \frac{AP_A}{AP_B}$$

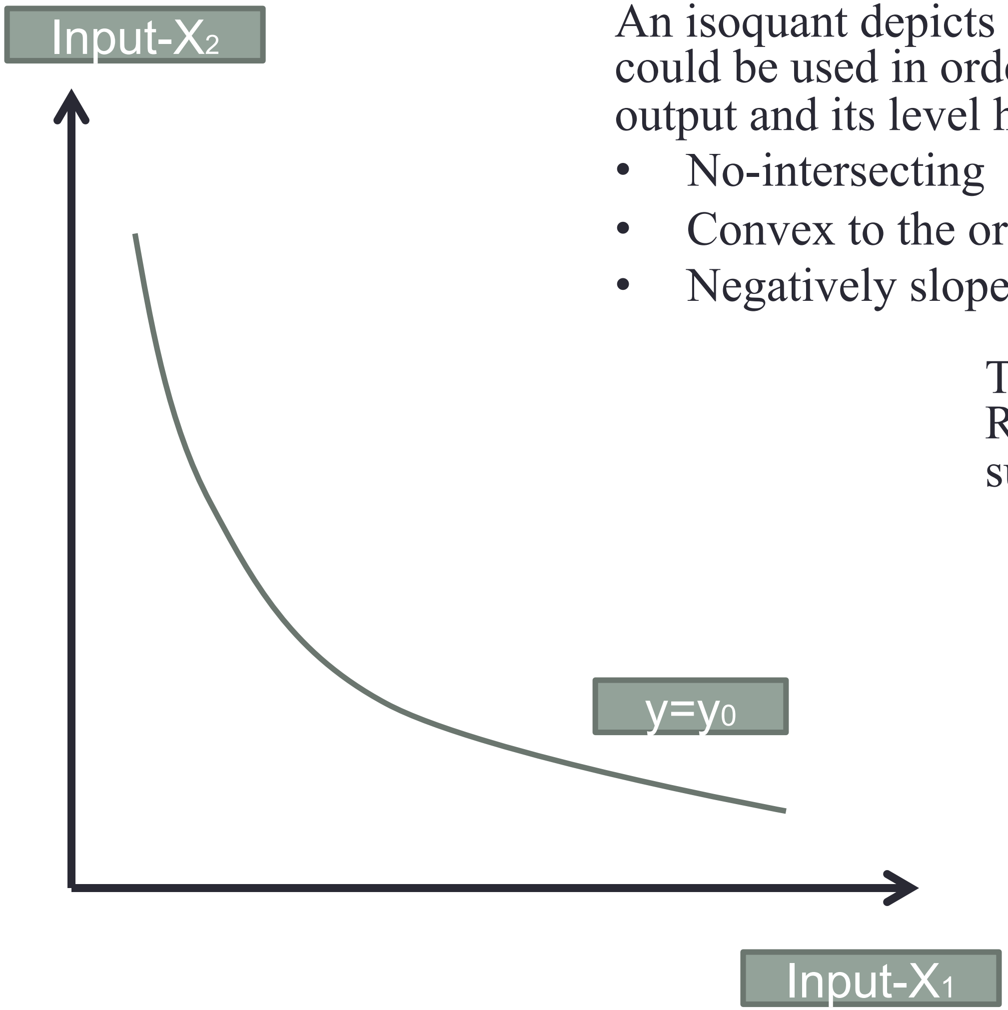
Which stages are inefficient? Why?

Long run Production Function

- In the long run the quantities of all inputs are variable.



LRPF and isoquants

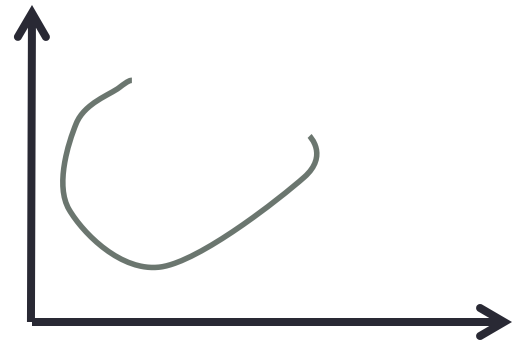


An isoquant depicts all combinations of labor and capital that could be used in order to produce a particular quantity of output and its level has an associated isoquant.

- No-intersecting
- Convex to the origin
- Negatively slope

The slope of the isoquant is called MRTS. Reflects the rate at which labor can be substituted for capital holding y constant.

$$MRTS = \frac{dx_2}{dx_1} = -\frac{MP_1}{MP_2}$$



Returns to scale I

- Returns to scale refers to the concept that describe the degree to which a proportional increase in all inputs increase-decrease output.

Returns to scale	Mathematical Definition	Definition
Constant-CRS	$f(a x_1, a x_2, \ldots, a x_n) = a f(x_1, x_2, \ldots, x_n)$	A proportional increase in all input(-s) results the same increase in output(-s)
Increasing-IRS	$f(a x_1, a x_2, \ldots, a x_n) > a f(x_1, x_2, \ldots, x_n)$	A proportional increase in all input(-s) results in a more increase in output(-s)
Decreasing-DRS	$f(a x_1, a x_2, \ldots, a x_n) < a f(x_1, x_2, \ldots, x_n)$	A proportional increase in all input(-s) results in a less increase in output(-s)

Returns to scale II

- Empirical analyses for RTS focus on:

1)Total elasticity of production or elasticity of scale (depends on non-homogeneity). Measures the proportional change in output resulting from a unit proportional increase in all inputs.

$$\varepsilon = \sum_{i=1}^{n} E_i = E_1 + E_2 + \ldots + E_n = \frac{\partial f(x_1, x_2, \ldots, x_{n-1})}{\partial x_{i-1}} \frac{x_{i-1}}{y} + \frac{\partial f(x_1, x_2, \ldots, x_{n-1})}{\partial x_{i-2}} \frac{x_{i-2}}{y} + \ldots + \frac{\partial f(x_1, x_2, \ldots, x_{n-1})}{\partial x_{i-n}} \frac{x_{i-n}}{y}$$

Returns to scale	Mathematical Definition	Definition	Total Elasticity of Production
Constant-CRS	$f(\mathrm{a}\,x_1, \mathrm{a}\,x_2, \ldots, \mathrm{a}\,x_n) = \mathrm{a}\, f(x_1, x_2, \ldots, x_n)$	A proportional increase in all input(-s) results the same increase in output(-s)	$\varepsilon = 1$
Increasing-IRS	$f(\mathrm{a}\,x_1, \mathrm{a}\,x_2, \ldots, \mathrm{a}\,x_n) > \mathrm{a}\, f(x_1, x_2, \ldots, x_n)$	A proportional increase in all input(-s) results in a more increase in output(-s)	$\varepsilon > 1$
Decreasing-DRS	$f(\mathrm{a}\,x_1, \mathrm{a}\,x_2, \ldots, \mathrm{a}\,x_n) < \mathrm{a}\, f(x_1, x_2, \ldots, x_n)$	A proportional increase in all input(-s) results in a less increase in output(-s)	$\varepsilon < 1$

Returns to scale III

- Empirical analyses for RTS focus on:

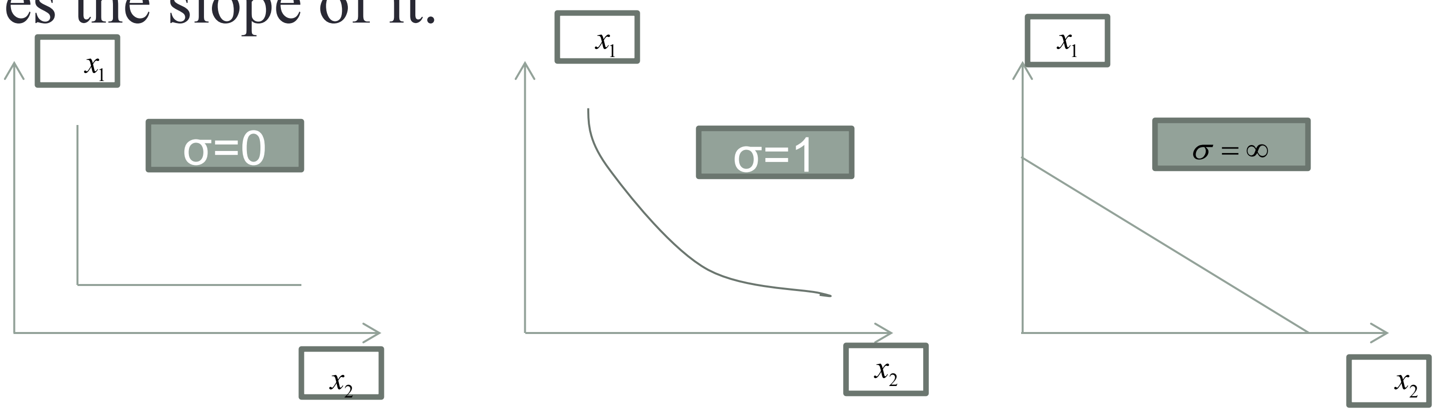
2) Partial production elasticities (can you talk about the signs?)

$$E_i = \frac{\partial f\left(x_1, x_2, \ldots, x_n\right)}{\partial x_i}\frac{x_i}{y}$$

Measures the proportional change in output resulting from a proportional increase in the i-th input with all the other input levels held constant.

3) Elasticity of substitution (closely related to MTRS). The elasticity of substitution measures the curvature of the isoquant. MRTS measures the slope of it.

$$E_i = \frac{\dfrac{d\left(x_2/x_1\right)}{\left(x_2/x_1\right)}}{\dfrac{d\left(MP_1/MP_2\right)}{\left(MP_1/MP_2\right)}}$$



An example

1. Let us assume the following production function $y = 2x_1^{0.5}x_2^{0.4}$

Can you calculate partial production elasticities, total production elasticity and elasticity of substitution;

2. For the previous production function we assume that

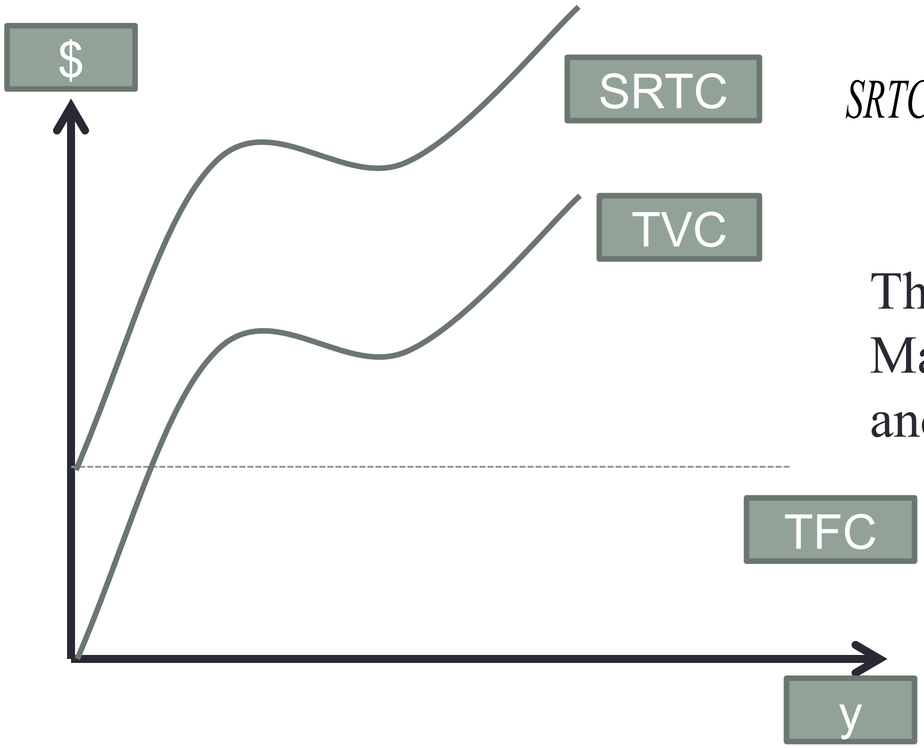
$TC = w_1x_1 + w_2x_2$ while $y=10$ can you please find the quantities of labor and capital that maximize the output;

Short and long run Cost-Profit

	Variable quantities	Fixed quantities
SR Cost minimization	Labor	Capital, output
LR Cost minimization	Labor, capital	Output
SR Profit maximization	Labor, output	Capital
LR Profit maximization	Labor, capital, output	-

Cost Minimization I

The SRTC is given as the following sum:



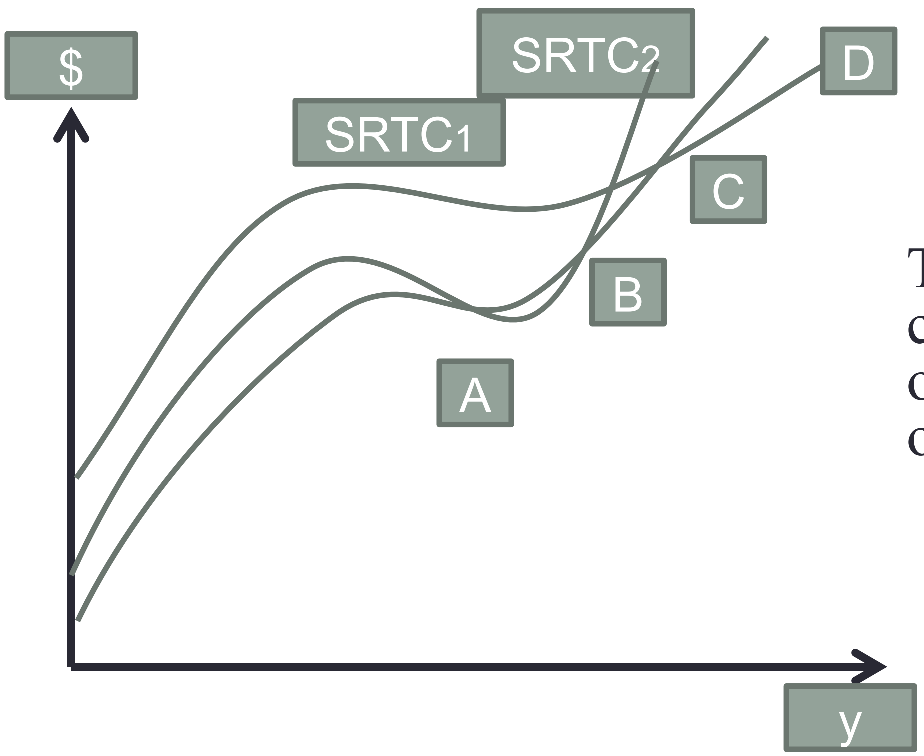
$$SRTC = TVC + TFC = w_1x_1 + w_2x_{20} = w_1 f^{-1}\left(y, x_2 = x_{20}\right) + w_2x_{20} = c(y)$$

The prices of output, labor and capital p, w_1, w_2

Making labor the subject we can have $f^{-1}(y, x_2 = x_{20})$

and $TVC = w_1 f^{-1}(y, x_2 = x_{20})$

Cost Minimization II



$$LRMC = \frac{\partial LRTC}{\partial y}, LRAC = \frac{LRTC}{y}$$

$$LRTC = ABCD$$

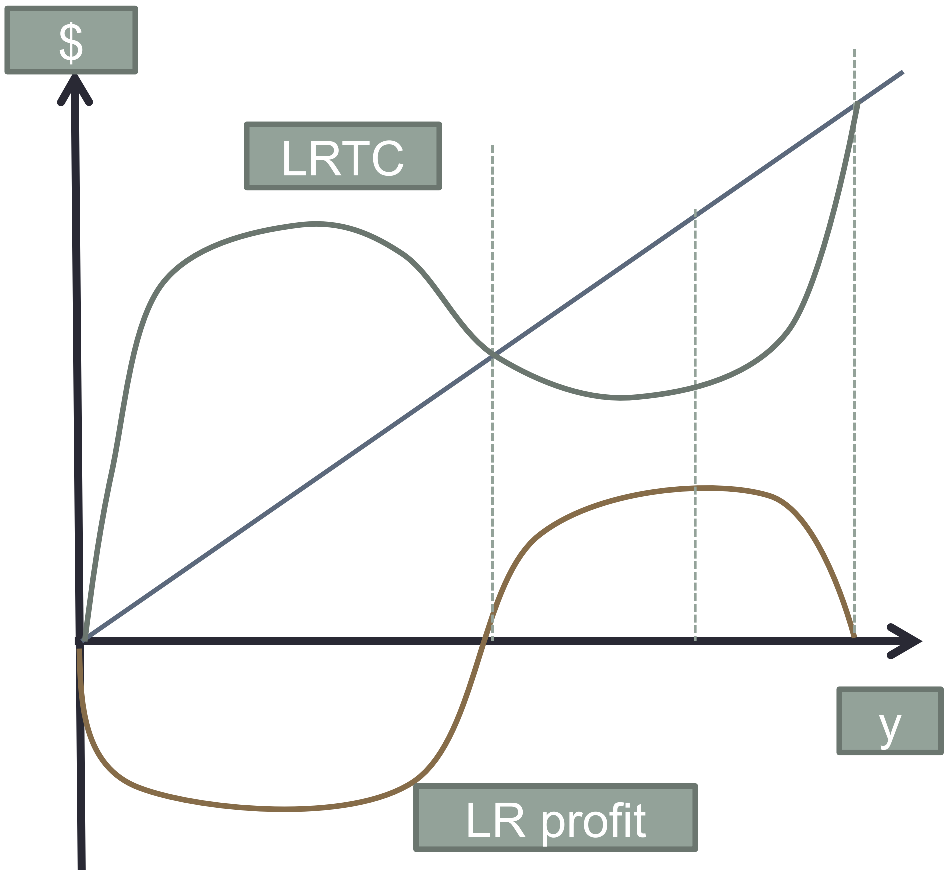
The segments of these curves corresponding of three different levels of capital and the point A,B,C,D trace out the LRTC curve.

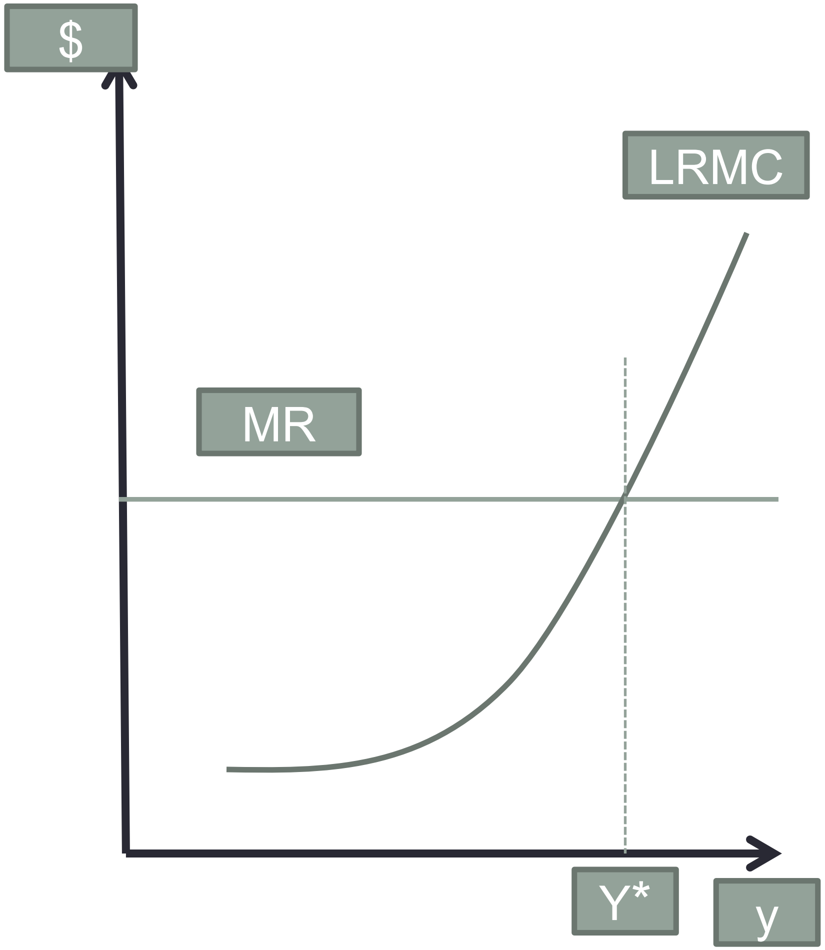
Let us assume a family of SRTC curves! (can we use isoquants?)

Profit Maximization I

- The profit of a firm is defined as the difference of total revenue from total costs. In perfect competition TR curve is a straight line with slope p. The LR profit curve is equal to TR-LRTC curve.
- Can we identify the LR profit maximizing level of output and how? MR=LRMC

Profit Maximization II





Econometric specification-introduction

The question of choosing an appropriate production function is dominant while a variety has been used in the literature. We assume a Cobb-Douglas (restrictions upon the production structure) specification as: $Q \equiv Y = Ax_1^{b_1}x_2^{b_2} \Leftrightarrow \ln Y = \ln A + b_1 \ln x_1 + b_2 \ln x_2$

Or a more complex as Translog

$$\ln Y = b_0 + b_1 \ln x_1 + b_2 \ln x_2 + \frac{1}{2}\left[b_{11}\left(\ln x_1\right)^2 + b_{22}\left(\ln x_2\right)^2\right] + b_{11} \ln x_1 \ln x_2$$

Or a CES $Y = A\left[bx_1^{-g} + (1-b)x_2^{-g}\right]^{-v/g}$

Duality in production I-Profit

- Define input demand and output supply equations curves
- Profit maximization. Remember that the profit-maximizing level of output its obtained by MR=LRMC
- How can we derive the output supply curve?

Hold input prices fixed, vary the output price repeatedly solving the MP obtaining for an output quantity for each price level $y^* = y^*(p, w_1, w_2)$

In a similar manner we can define input demand curves for labor and capital $x_1^* = x_1^*(p, w_1, w_2), x_2^* = x_2^*(p, w_1, w_2)$

$y = f(x_1, x_2) =$
$f(x_1^*(p, w_1, w_2), x_2^*(p, w_1, w_2))$

Profit maximization can be obtained solving the following:

$$\pi = TR - TC = py - (w_1 x_1 + w_2 x_2) = pf(x_1, x_2) - (w_1 x_1 + w_2 x_2)$$

$$\frac{\partial \pi}{\partial x_1} = 0, \frac{\partial \pi}{\partial x_2} = 0$$

Duality in production II-Cost

- Can we choose cost minimization against profit maximization and why?
- First of all we can form $TC = (w_1 x_1 + w_2 x_2)$ and use the Lagrange function $L = w_1 x_1 + w_2 x_2 - \lambda\left(y - f\left(x_1, x_2\right)\right)$ in order to be $\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x_2} = 0, \frac{\partial L}{\partial \lambda} = 0$

  solved and derive the input demand functions

  $x_1^c = x_1^c\left(p, w_1, w_2\right), x_2^c = x_2^c\left(p, w_1, w_2\right)$
- Where is the duality?

Duality in production III

- Primal approach: The knowledge of production function leads to output and input demand equations.
- Dual approach: Deriving output supply and input demand equations directly from an estimated profit or cost function.
- Is duality useful? Why?

Hotelling's Lemma

$$\pi = p\left[y(p,w_1,w_2)\right]-\left\{w_1\left[x_1(p,w_1,w_2)\right]+w_2\left[x_2(p,w_1,w_2)\right]\right\} \equiv \pi(p,w_1,w_2)$$

- Hotelling's Lemma states that the first partial derivatives of the profit function with respect to each of the input prices define the negative of the input demand functions.
- It is also assumes that $\frac{\partial \pi}{\partial w_1} = -x_1^*(p,w_1,w_2), \frac{\partial \pi}{\partial w_2} = -x_2^*(p,w_1,w_2)$
- Using the symmetry (Young's theorem we have)

$$\frac{\partial \pi}{\partial p} = y^*(p,w_1,w_2)$$

Shepard's Lemma

- A cost function is defined as the minimum cost of producing a particular output with given input prices. In a similar manner we can have:

$$\pi = p\left[y(p, w_1, w_2)\right] - \left\{w_1\left[x_1(p, w_1, w_2)\right] + w_2\left[x_2(p, w_1, w_2)\right]\right\} \equiv \pi(p, w_1, w_2)$$

- Shepard's lemma (envelop theorem) states that the first partial derivative w.r.t. to each of the input prices defines the conditional input demand function

- Symmetry: $\frac{\partial c}{\partial w_1} = x_1^c(p, w_1, w_2), \frac{\partial c}{\partial w_2} = x_2^c(p, w_1, w_2)$

Properties of Profit-Cost functions

- Profit Function

1. $\pi^*(p,w) \geq 0$, for $p, w \geq 0$

2. $\pi^*(p^a,w) \geq \pi^*(p^b,w)$, for $p^a \geq p^b$

3. $\pi^*(p,w^a) \leq \pi^*(p,w^b)$, for $w^a \geq w^b$

4. $\pi^*(p,w)$ homogeneous of degree 1 in all p

5. $\frac{\partial \pi^*(p,w)}{\partial p}, \frac{\partial \pi^*(p,w)}{\partial w_i}$ homogeneous of degree 0 in all p

6. $\pi^*(p,w)$ is convex in all prices if y=f(x) is strictly concave

- Cost Function

1. $c^*(y,w) \geq 0$, for $y, w \geq 0$

2. $c^*(y,w^a) \geq c^*(y,w^b)$, for $w^a \geq w^b$

3. $c^*(y,w)$ homogeneous of degree 1 in all p

4. $\frac{\partial c^*(y,w)}{\partial w_i}$ homogeneous of degree 0 in all p

5. $c^*(y,w)$ is weakly convex in all prices if y=f(x) is strictly quasi-concave

Technical change in a production function I

- In the case where our problem has time the Cobb-Douglas function is becoming as $\ln Y = b_0 + b_1 \ln x_1 + b_2 \ln x_2 + b_t t, t = 1, 2, .., T$

with b_t to estimate the annual percentage change in output resulting from technological change

- Hicks neutral technical change (all the isoquants are shifting each year with any change in their shape).

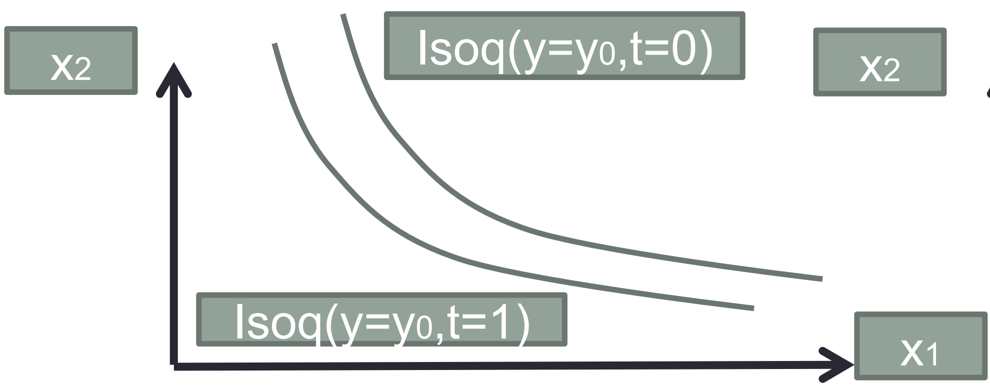
Technical change in a production function II

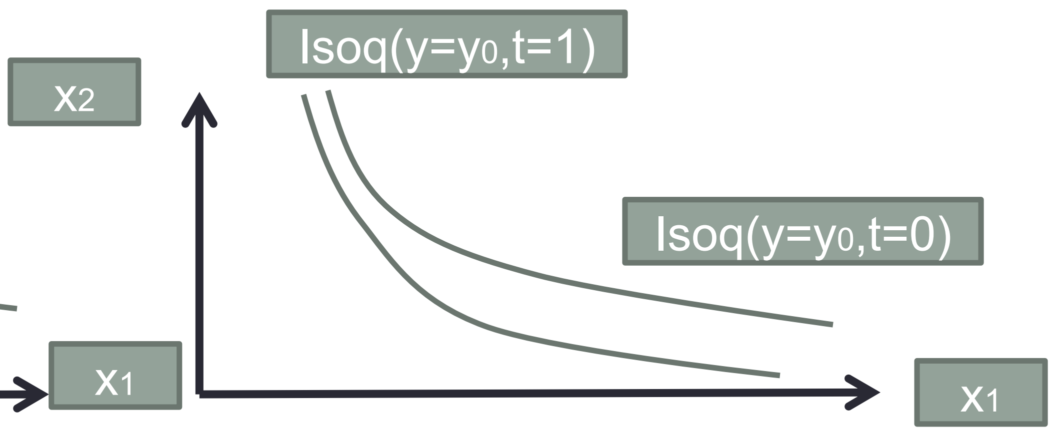
- In the case of a cost translog function we have:

$$\ln TC = b_0 + b_1 \ln w_1 + b_2 \ln w_2 + b_3 \ln y + b_{12} \ln w_1 \ln w_2 + b_{13} \ln w_1 \ln y + b_{23} \ln w_2 \ln y +$$

$$+ \frac{1}{2}\left[b_{11} (\ln w_1)^2 + b_{22} (\ln w_2)^2 + b_{33} (\ln y)^2 \right] + b_t t + b_{tt} t^2 + v, t = 1, 2, .., T$$

- What the partial derivative of cost w.r.t to time is telling us?
- Hicks neutral technical change again!





Technical change in a production function III

Non-neutral technical change (biased technical change?

$$\ln TC = b_0 + b_1 \ln w_1 + b_2 \ln w_2 + b_3 \ln y + b_{12} \ln w_1 \ln w_2 + b_{13} \ln w_1 \ln y + b_{23} \ln w_2 \ln y +$$

$$+ \frac{1}{2}\left[b_{11} \left(\ln w_1 \right)^2 + b_{22} \left(\ln w_2 \right)^2 + b_{33} \left(\ln y \right)^2 \right] + b_{1t} \ln w_1 t + b_{2t} \ln w_2 t + b_{3t} \ln y\, t + b_t t + b_{tt}\, t^2 + v, t = 1, 2, .., T$$

Can you identify and explain the differences from Hicks neutral technical change?

Production Technology I

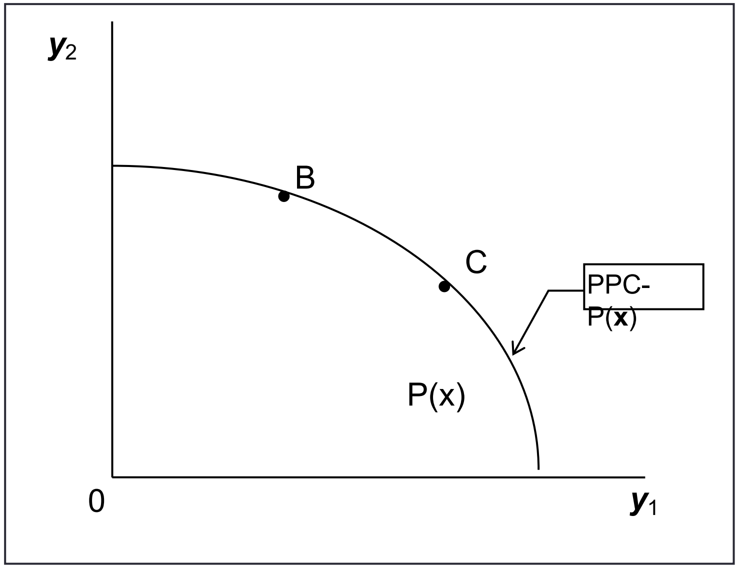
- We assume that there is a production technology that allows ***transformation*** of a ***vector of inputs*** into a ***vector of outputs***
  $S = \{(\mathbf{x},\mathbf{y}): \mathbf{x} \text{ can produce } \mathbf{y}\}$.
- In most of the cases a multi-input, multi-output technology is considered
- Technology set is assumed to satisfy some basic axioms.
- It can be equivalently represented by
  - **Output sets**
  - **Input sets**
  - **Output and input distance functions**
- A production function provides a relationship between the maximum feasible output (in the single output case) for a given set of input
- Single output/single input; single output/multiple inputs; multi-output/multi-input

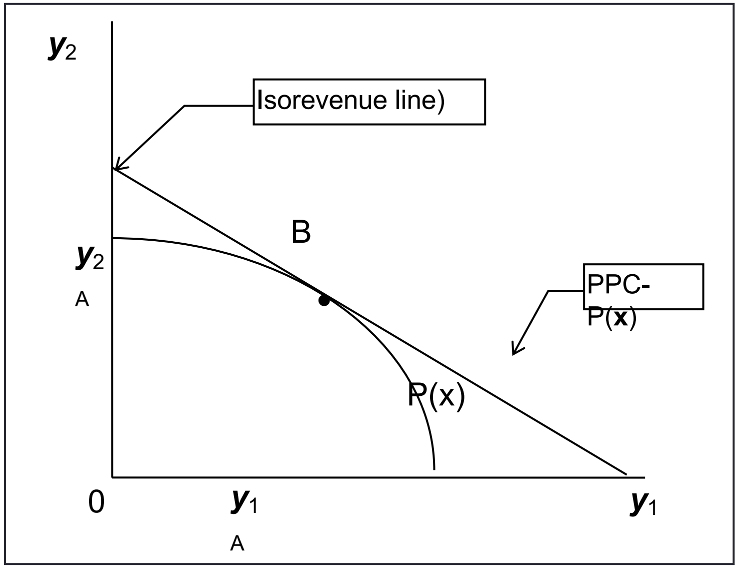
Production Technology II

- To present and conceptualize a multi input, multi-output technology set is very difficult.
- Thus we can have a single input as a function of two outputs $x_1 = g(y_1, y_2)$

In order to define the production possibility set (PPC)! What is presenting?

- The revenue equivalent to the isocost line is the isorevenue line with slope $-p_1/p_2$





Output set

- Output set P(x) for a given vector of inputs, x, is the set of all possible output vectors q that can be produced by x.

$$P(x) = \{q: x \text{ can produce } q\} = \{q : (x,q) \in S\}$$

  - P(x) satisfies a number of intuitive properties Boundary of P(x) is the production possibility curve

1. $P(x)$ is closed

2. $P(x)$ is bounded

3. $P(x)$ is convex

4. $0 \in P(x)$

5. Non-zero values cannot be produced from 0 level of inputs

6. $P(x)$ satisfies strong disposability of inputs

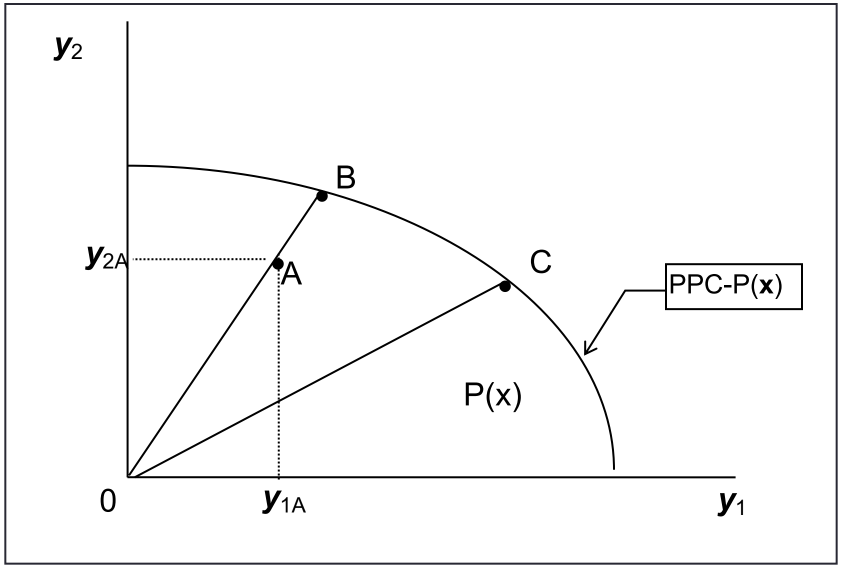
Output Distance Function

- Output distance function for two vectors **x** (input) and **q** (output) vectors, the output distance function is defined as:

  $d_o(\mathbf{x},\mathbf{q}) = \min\{\delta: (\mathbf{q}/\delta) \in P(\mathbf{x})\}$ Farell (Shepard???)

- Properties:
  - Non-negative
  - Non-decreasing in **q**; non-increasing in **x**
  - Linearly homogeneous in **q**
  - if **q** belongs to the production possibility set of **x** (i.e., $\mathbf{q} \in P(\mathbf{x})$), then $d_o(\mathbf{x},\mathbf{q}) \leq 1$ and the distance is equal to 1 only if q is on the frontier.

Output Distance Function



$D_o(x,\mathbf{y})$

The value of the distance function is equal to the ratio δ=0A/0B.

Output-oriented Technical Efficiency Measure:

$$TE = 0A/0B = d_o(x,q)$$

Output-Input set

- Output set P(x) for a given vector of inputs, x, is the set of all possible output vectors q that can be produced by x.

  $$P(x) = \{q: x \text{ can produce } q\} = \{q : (x,q) \in S\}$$

  - P(x) satisfies a number of intuitive properties including: nothing can be produced from x; set is closed, bounded and convex
  - Boundary of P(x) is the production possibility curve
- An Input set L(q) can be similarly defined as set of all input vectors x that can produce q.

  $$L(q) = \{x: x \text{ can produce } q\} = \{x: (x , q) \in S\}$$

  - L(q) satisfies a number of important properties that include: closed and convex
  - Boundary of L(q) is the isoquant curve
- These sets are used in defining the input and output distance functions

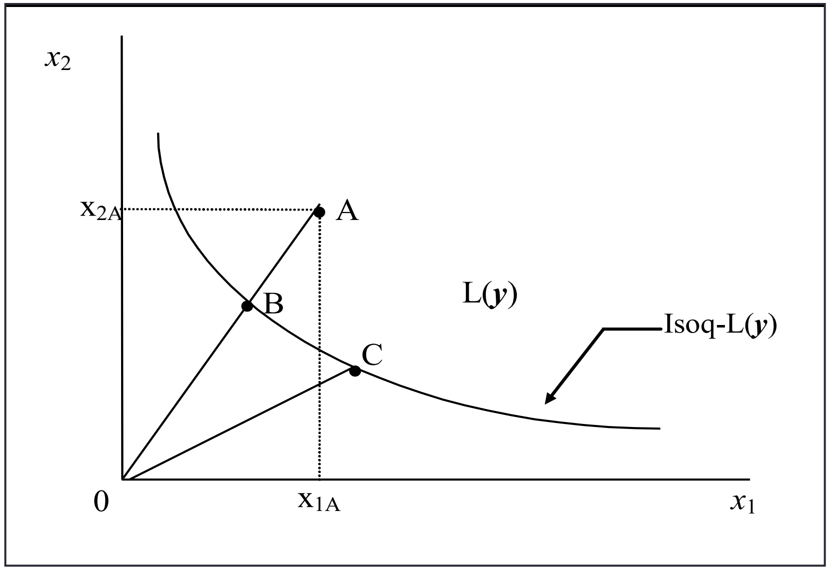
Input Distance Function

- Input distance function for two vectors **x** (input) and **q** (output) vectors is defined as:

  $d_i(\mathbf{x},\mathbf{q}) = \max\{\rho: (\mathbf{x}/\rho) \in L(\mathbf{q})\}$

- Properties:
  - Non-negative
  - Non-decreasing in **x**; non-increasing in **q**
  - Linearly homogeneous in **x**
  - if **x** belongs to the input set of **q** (i.e., $\mathbf{x} \in L(\mathbf{q})$), then $d_i(\mathbf{x},\mathbf{q}) \geq 1$ and the distance is equal to 1 only if **x** is on the frontier

Input Distance Function



$D_i(x,y$

The value of the distance function is equal to the ratio ρ=0A/0B.

Technical Efficiency = TE = 1/d_i(x,q) = OB/OA

Profit Maximization

- Firms produce a vector of M outputs (q) using a vector of K inputs (x)
- The production technology (set) is:

- Maximum profit is defined as:

where p is a vector of M output prices and w is a vector of K input prices

References

1. S.C. Kumbhakar and C.A. Knox Lovell (2000) *Stochastic Frontier Analysis* (Cambridge University Press) ISBN: 0521481848
2. Greene W H (1998) Frontier Production Functions in H. Peasaran and P. Schmidt (eds) *Handbook of Applied Econometrics, volume 2, microeconomics* (Blackwells)
3. An Introduction to Efficiency and Productivity Analysis (2nd Ed.) *Coelli, Rao, O'Donnell and Battese* Springer, 2005

Supplemented with material from other published papers relative to these issues

Appendix-First Problem

The output of a firm is given w.r.t to time of operation from the following equation: $Q(t)=\frac{t^3}{30}-\frac{t^2}{5}+\frac{3t}{10}+120$

Can you identify the time where the production will be optimal?

Solution: First we have to find the first derivative as

$$\frac{dQ(t)}{dt}=\frac{3t^2}{30}-\frac{2t}{5}+\frac{3}{10}=0\Leftrightarrow \ldots.\Leftrightarrow$$

$$(t-1)(t-3)=0\Leftrightarrow t=1 \text{ or } t=3$$

Then calculate $\frac{d^2Q(t)}{dt^2}=\frac{6t}{30}-\frac{2}{5}$. For t=1 $\frac{d^2Q(t)}{dt^2}=-0.2<0$ while for t=3

$$\frac{d^2Q(t)}{dt^2}=0.2>0$$

maximum----minimum

Second Problem

A competitive firm produce two goods X,Y with prices 54 and 52 euros. The total cost function is given as follows: $TC = 3X^2 + 3XY + 2Y^2 - 100$

Can you specify the corresponding quantities that maximize firm's profit?

Solution of second problem I

First we have to define the profit function of the firm as follows:

$$\Pi = TR - TC = 54X + 52Y - 3X^2 - 3XY - 2Y^2 + 100$$

Then we have to differentiate the profit function w.r.t. to X,Y.

$$\frac{\partial \Pi}{\partial X} = \frac{\partial\left(54X + 52Y - 3X^2 - 3XY - 2Y^2 + 100\right)}{\partial X} = 54 - 6X - 3Y$$

$$\frac{\partial \Pi}{\partial Y} = \frac{\partial\left(54X + 52Y - 3X^2 - 3XY - 2Y^2 + 100\right)}{\partial Y} = 52 - 3X - 4Y$$

$$\frac{\partial^2 \Pi}{\partial X^2} = -6$$

$$\frac{\partial^2 \Pi}{\partial Y^2} = -4$$

$$\frac{\partial^2 \Pi}{\partial X \partial Y} = -3$$

Solution of second problem II

Setting the first derivatives equal to zero we can specify the quantities that may maximize the profit function.

$$\frac{\partial \Pi}{\partial X} = \frac{\partial\left(54X + 52Y - 3X^2 - 3XY - 2Y^2 + 100\right)}{\partial X} = 54 - 6X - 3Y = 0$$

$$\frac{\partial \Pi}{\partial Y} = \frac{\partial\left(54X + 52Y - 3X^2 - 3XY - 2Y^2 + 100\right)}{\partial Y} = 52 - 3X - 4Y = 0$$

Thus, $X = 4, Y = 10$. Then we can calculate the following magnitude

$$\Delta = \Pi_{XX}\Pi_{YY} - \Pi_{XY}^2 = 15$$

Is there a maximum??

Some Mathematics-Langrange

- Third Problem

A firm's production technology can be specified by the following Cobb-Douglas production function $Q = 10K^{0.5}L^{0.5}$. What are the cost-minimizing quantities of its two inputs capital and labor if the firm wishes to produce an output of 500 units given that the wage rate is 8 and the price of capital is 2?

Solution of third problem I

Setting up the langragian function we have:

$$LA = p_k K + p_l l + \lambda\left(500 - 10K^{0.5}L^{0.5}\right)$$

Theory suggests that we have to differentiate partially with respect to capital, labor and λ taking the first-order conditions for a minimum. Thus:

$$\frac{\partial LA}{\partial \mathrm{K}} = 0 \Leftrightarrow \frac{\partial p_k K + p_l l + \lambda\left(500 - 10K^{0.5}L^{0.5}\right)}{\partial \mathrm{K}} = 0 \Leftrightarrow 2 - 5\lambda K^{-0.5}L^{0.5} = 0$$

$$\frac{\partial LA}{\partial L} = 0 \Leftrightarrow \frac{\partial p_k K + p_l l + \lambda\left(500 - 10K^{0.5}L^{0.5}\right)}{\partial L} = 0 \Leftrightarrow 8 - 5\lambda K^{0.5}L^{-0.5} = 0$$

$$\frac{\partial LA}{\partial \lambda} = 0 \Leftrightarrow \frac{\partial p_k K + p_l l + \lambda\left(500 - 10K^{0.5}L^{0.5}\right)}{\partial \lambda} = 0 \Leftrightarrow 500 - 10K^{0.5}L^{0.5} = 0$$

From which we have that $\frac{5\lambda K^{-0.5}L^{0.5}}{5\lambda K^{0.5}L^{-0.5}} = \frac{2}{8} \Leftrightarrow \mathrm{K} = 4L$ and therefore $\mathrm{K} = 100, L = 25$ and $\lambda = 0.8$

Solution of third problem II

In order to see whether costs are indeed minimized at this input combination, we need to check the second-order conditions for a minimum are satisfied at the point (100,25,0.8)

$$\frac{\partial^2 LA}{\partial K^2} = 2.5\lambda K^{-1.5} L^{0.5}, \frac{\partial^2 LA}{\partial L^2} = 2.5\lambda K^{0.5} L^{-1.5}, \frac{\partial^2 LA}{\partial K \partial L} = -2.5K^{-0.5} L^{-0.5}$$

and the partial derivatives of the constraint w.r.t to K,L are

$$\frac{\partial Q}{\partial K} = 5K^{-0.5} L^{0.5}, \frac{\partial Q}{\partial L} = 5K^{0.5} L^{-0.5}$$

Evaluated the above derivatives at the point (100,25,0.8) , the bordered Hessian determinant is given by:

$$H = \begin{vmatrix} 0 & 2.5 & 10 \\ 2.5 & 0.01 & -0.04 \\ 10 & -0.04 & 0.16 \end{vmatrix} = -4 < 0$$

$$H = \begin{vmatrix} f_{11} & f_{12} & f_{1n} \\ f_{21} & f_{22} & f_{2n} \\ f_{n1} & f_{n2} & f_{nn} \end{vmatrix}$$

The minimum cost of producing 50 units of output is 400, with the firm employing 100 units of capital and 25 of labor.

More than one constraints (Kuhn-Tucker)

Consider the following minimization problem:

$$\min f(x_1, x_2, .., x_n)$$
$$s.t\ g^1(x_1, x_2, .., x_n) \geq c_i, i = 1, 2, .., m$$
$$x_j \geq 0, \text{for all j}$$

The Langragean function is given us $\mathrm{L} = f(x_1, x_2, .., x_n) + \sum_{i=1} y_i \left(c_i - g^1(x_1, x_2, .., x_n)\right)$

If we differentiate the Langrangean function with respect to x,y we obtain the following set of first-order conditions:

$$\frac{\partial \mathrm{L}}{\partial x_j} = f_j - \sum_{i=1}^{m} y_i g_j^i \geq 0, x_j \geq 0, x_j \frac{\partial \mathrm{L}}{\partial x_j} = 0$$
$$\frac{\partial \mathrm{L}}{\partial x_j} = c_i - g^1(x_1, x_2, .., x_n) \leq 0, \mathrm{y}_i \geq 0, \mathrm{y}_i \frac{\partial \mathrm{L}}{\partial y_i} = 0$$

Partial derivative of the objective function w.r.t. to xj

Partial derivative of the i-th constraint w.r.t. to xj

Fourth Problem

A utility-maximising consumer's utility function takes the following form: $U(x) = 20x_1 - 2x_1^2 + 100x_2 - x_2^2$ where xi is the amount consumed of good i. Throughout, the prices of the two goods are held constant with the price of good I being 10 and the price of good 2 being unity. The consumer's money income is M. Can you find the quantities of the two goods that maximize the utility function?

Solution of fourth problem I

The Langragean function is given as $L = 20x_1 - 2x_1^2 + 100x_2 - x_2^2 + \lambda(\mathrm{M} - 10x_1 - x_2)$

Differentiating we will have:

$$\frac{\partial \mathrm{L}}{\partial x_1} = 20 - 4x_1 - 10\lambda \leq 0, x_1 \geq 0, x_1 \frac{\partial \mathrm{L}}{\partial x_1} = 0$$

$$\frac{\partial \mathrm{L}}{\partial x_2} = 100 - 2x_2 - \lambda \leq 0, x_2 \geq 0, x_2 \frac{\partial \mathrm{L}}{\partial x_2} = 0$$

$$\frac{\partial \mathrm{L}}{\partial xl} = \mathrm{M} - 10x_1 - x_2 \geq 0, \lambda \geq 0, \lambda \frac{\partial \mathrm{L}}{\partial \lambda} = 0$$

Let us now see how consumption of the two goods will vary as the consumer's income changes. We first consider the circumstances under the consumer will not consume the first good and λ=2.

If the marginal utility of spending a euro on good 1 when 0 units are consumed is less than the marginal utility of money income as measured by the Langragean multiplier (only positive if all income is spent), it is not in the consumer's interest to consume good 1.

Solution of fourth problem II

To satisfy the inequality on λ, x2 must be positive and not greater than 49.Given that the price of the second good is unity we can have:

$x_1 = 0, x_2 = \mathrm{M}, 0 \le \mathrm{M} \le 49$

If both goods are consumed and there is no slack in the budget constraint then we must have

$$\begin{aligned} &20 - 4x_1 - 10\lambda = 0 \\ &100 - 2x_2 - \lambda = 0 \\ &\mathrm{M} - 10x_1 - x_2 = 0 \end{aligned}$$

and solving

$$x_1 = \frac{20M - 980}{204}, x_2 = \frac{4M + 9800}{204},$$
$$\lambda > 0, \lambda = 20 - 4x_1 = 100 - 2x_2$$

thus x1 can't exceed 5,x2 can't exceed 50, M will be greater that 49 and less than 100.For M=100, λ is 0 and increases in income beyond this point lead to no changes in the quantities consumed of the 2 goods. Hence, $x_1 = 5, x_2 = 50, 100 \le \mathrm{M}$.

Solution of fourth problem III

We can also investigate how the values of the lagrangean multiplier varies as income increases.

1. For $0 \le M \le 49$ only the second god is consumed with $x_1 = 0, x_2 = M$,
2. For $49 \le M \le 100$ then $x_2 = \dfrac{4M + 9800}{204}, \lambda = 100 - 2x_2$
3. For $M > 100, x_2 = 50, \lambda = 0$

The maximum value function

$$V(M) = 100M - M^2, 0 < M \le 49$$
$$V(M) = \frac{480200 + 800M - 4M^2}{204}, 49 < M \le 100$$
$$V(M) = 2550, M > 100$$

$$\frac{dV(M)}{dM} = 100 - 2M, 0 < M \le 49$$
$$\frac{dV(M)}{dM} = \frac{800 - 8M}{204}, 49 < M \le 100$$
$$\frac{dV(M)}{dM} = 0, M > 100$$

Two Examples

Please solve the following

$$\min \ 2x_1^2 - 32x_1 + x_2^2 - 20x_2$$
$$s.t \ x_1 + 2x_2 \geq 36$$
$$3x_1 + x_2 \geq 43$$
$$x_j \geq 0, \text{for j=1,2}$$

$$\max \ 24x_1 - x_1^2 - x_1x_2 - x_2^2$$
$$s.t \ \text{-}\left(5 - x_1\right)^3 + x_2 \leq 0$$
$$x_j \geq 0, \text{for j=1,2}$$

Linear Programming

Please solve the following linear problem.

$$\max z = 6x_1 + 7x_2$$

$$s.t \quad 2x_1 + 3x_2 + x_3 = 12$$

$$2x_1 + x_2 + x_4 = 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Initial Tableau for Simplex-Slacks

What are slacks?

		c_j	**6**	**7**	**0**	**0**	**0**
x_B	c_b	b	x_1	x_2	x_3	x_4	θ
x_3	0	12	2	3	1	0	12
x_4	0	8	2	1	0	1	8
	z						

Solution using Simplex method I

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_3	0	12	2	3	1	0	k_6
	0	8	2	1	0	1	k_7
	z	k_1	k_2	k_3	k_4	k_5	

Calculating k's

$k_j = c_j - c_b * b$

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_3	0	12	2	3	1	0	k_6
	0	8	2	1	0	1	k_7
	z	k_1	$k_2 = 6-(2\cdot 0+2\cdot 0) = 6$	$k_3 = 7-(3\cdot 0+1\cdot 0) = 7$	$k_4 = 0$	$k_5 = 0$	

Solution using Simplex method II

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_3	0	12	2	3	1	0	k_6
	0	8	2	1	0	1	k_7
	z	0	6	7	0	0	

Finding pivot element

Calculating θ's

$$k_6 = 12/3 = 4, k_7 = \frac{8}{1} = 8$$

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_3	0	12	2	3	1	0	4
x_4	0	8	2	1	0	1	8
	z	0	6	7	0	0	

Solution using Simplex method III

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7						
x_4	0						
	z						

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	4	2/3	1	1/3	0	
x_4	0						
	z						

Here the key point to do same work with matrices: for example: Divide with 1/3.

Solution using Simplex method IV

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	4	1/3	1	1/3	0	k_6
x_4	0	4	4/3	0	-1/3	1	k_7
	z	k_1	k_2	k_3	k_4	k_5	

Do some work with linear matrices.

Now we choose x_1.

Continue with the same logic.

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	4	2/3	1	1/3	0	k_6
x_4	0	4	4/3	0	-1/3	1	k_7
	z	28	4/3	0	-7/3	0	

Solution using Simplex method V

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	4	2/3	1	1/3	0	6
x_4	0	4	4/3	0	-1/3	1	3
	z	28	4/3	0	-7/3	0	

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	4	2/3	1	1/3	0	-
x_1	6	3	**1**	0	-1/4	3/4	-
	z	k_1	k_2	k_3	k_4	k_5	

Solution using Simplex method VI

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	2	0	1	1/2	- 1/2	-
x_1	6	3	**1**	0	-1/4	3/4	-
	z	k_1	k_2	k_3	k_4	k_5	

Finalize the procedure when we have values equal or less than zero.

		c_j	**6**	**7**	**0**	**0**	
x_B	c_B	b	x_1	x_2	x_3	x_4	θ
x_2	7	2	0	1	1/2	-1/2	-
x_1	6	3	**1**	0	-1/4	3/4	-
	z	32	0	0	-2	-1	