

Prelude

Stationarity. A time series $\{X_t\}$ is said to be (*weakly*) *stationary* if its first two moments do not depend on time. Formally,

$$E(X_t) = \mu, \forall t,$$

$$Var(X_t) = E[(X_t - \mu)^2] = \sigma^2, \forall t,$$

$$Cov(X_t, X_{t-k}) = \gamma_k, \forall t \text{ and for each positive lag } k.$$

That is, the mean and variance are constant over time, and the covariance between X_t and X_{t-k} depends only on the lag k , not on the specific time t .

Common types of non-stationarity in macroeconomic data. Macroeconomic time series are often non-stationary, typically for one of two reasons:

1. **Trend stationarity (deterministic trend).** The series fluctuates around a deterministic function of time, such as

$$X_t = a + bt + u_t,$$

where u_t is stationary. Removing the deterministic trend (e.g. by regression on t or by detrending) yields a stationary series. Such a series is called *trend stationary*.

2. **Stochastic trend or unit root non-stationarity (difference stationarity).** The series contains a stochastic trend and can be written as

$$X_t = X_{t-1} + u_t,$$

where u_t is stationary. Here, shocks have permanent effects, and the level of the series follows a random walk. The first difference

$$\Delta X_t = X_t - X_{t-1} = u_t$$

is stationary, so X_t is said to be *integrated of order one*, or $I(1)$.

In macroeconomic data, it is common to encounter both deterministic and stochastic trends, and distinguishing between them is crucial for correct modeling and inference.

FRED Growth Rate Formulas

1. Change

Compute the differences between consecutive observations. This is known as **differencing**. In the time series jargon: the **first Differences** of X_t with $t = 1, 2, \dots, T$ are obtained through

$$\Delta X_t = X_t - X_{t-1}$$

Differencing can help stabilize the mean of a time series by removing changes in the level of a time series, and therefore eliminating (or reducing) trend and seasonality.¹

Differencing is often used to achieve **stationarity** by working with changes rather than levels.

Linear trend example (deterministic no disturbances). Let

$$X_t = a + bt,$$

with intercept a and slope b . The first difference is

$$\Delta X_t \equiv X_t - X_{t-1} = (a + bt) - [a + b(t-1)] = b,$$

which removes the linear trend (the differenced series has constant mean b).

Quarterly seasonality. For quarterly data with seasonal pattern, use the seasonal difference

$$\Delta_4 X_t \equiv X_t - X_{t-4},$$

which eliminates effects that repeat every four periods.

Both trend and seasonality. The term **seasonal type differencing** applies to $\Delta_k X_t = X_t - X_{t-k}$ and it reduces seasonality. In quarterly series use $k = 4$ whereas in monthly series use $k = 12$. If both (non-stationarity and seasonality) are present, apply both operators (in either order):

$$\Delta \Delta_4 X_t = \Delta_4 \Delta X_t = (X_t - X_{t-1}) - (X_{t-4} - X_{t-5}) = X_t - X_{t-1} - X_{t-4} + X_{t-5}$$

If the differenced data do not appear to be stationary (you observe “trends”) then it may be necessary to difference the data a second time to obtain a **stationary** series:

$$\Delta^2 X_t = \Delta \Delta X_t = \Delta (X_t - X_{t-1}) = (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) = X_t - 2X_{t-1} + X_{t-2}$$

The “ k -th difference” operator: $\underbrace{\Delta \Delta \cdots \Delta}_{k \text{ times}} X_t = \Delta^k X_t$ is defined as the application of Δ k times.

How much differencing? For macroeconomic time series, *first differencing is usually sufficient*. It is *rare* to need second differencing (let alone higher orders), and over-differencing can be harmful: it inflates noise, may introduce a negative autocorrelation and can obscure meaningful dynamics.

In practice:

¹From <https://otexts.com/fpp2/>

- Prefer a single difference $\Delta X_t = X_t - X_{t-1}$ to remove a stochastic linear trend.
- If seasonality is present, use a *seasonal* difference instead/as well, e.g. $\Delta_k X_t = X_t - X_{t-k}$ (for quarterly data $k = 4$), rather than applying a second nonseasonal difference.
- Consider deterministic terms (intercept, drift, linear trend, seasonal dummies) before adding another order of differencing.
- Diagnose with unit-root tests (e.g. ADF/PP), and inspect the ACF/PACF of levels and first-differenced data. Only if the series appears $I(2)$ should $\Delta^2 X_t$ be considered—and this is uncommon in macro data.

Rule of thumb. Start with ΔX_t (and Δ_s if seasonal). Move to $\Delta^2 X_t$ only with strong evidence of $I(2)$ behavior; otherwise, model the first-differenced series with appropriate ARMA terms and deterministic components.

2. Change from Year Ago

Lag- k differences. A general way to measure the change in a time series over k periods is through the *lag- k difference*, defined as

$$\Delta_k X_t = X_t - X_{t-k}$$

This represents the absolute change between the current observation and the one k periods earlier.

For example:

- For **quarterly data**, the *change from a year ago* corresponds to a lag of $k = 4$:

$$\Delta_4 X_t = X_t - X_{t-4}$$

- For **monthly data**, the *change from a year ago* uses a lag of $k = 12$:

$$\Delta_{12} X_t = X_t - X_{t-12}$$

- For **annual data**, the lag-1 difference

$$\Delta_1 X_t = X_t - X_{t-1}$$

simply measures the change from one year to the next — the usual first difference.

3. Percent Change

Percentage changes. Instead of using differences in levels, analysts often express changes as *percentage rates of change*, which provide a scale-free measure of growth. For small changes, the approximate percentage change is given by

$$100 \times \frac{\Delta_k X_t}{X_{t-k}}$$

This expresses how much X_t has increased (or decreased) relative to its value k periods earlier.

$$\% \Delta X_t = 100 \times \frac{X_t - X_{t-1}}{X_{t-1}}$$

Common terminology. In macroeconomic and financial reports, several standard abbreviations are used:

- **m-o-m** (or MoM): *month-on-month* change, i.e. the percentage change from one month to the next.
- **q-o-q** (or QoQ): *quarter-on-quarter* change, i.e. the percentage change from one quarter to the previous quarter.
- **y-o-y** (or YoY): *year-on-year* change, i.e. the percentage change compared with the same month or quarter a year earlier.

These transformations are widely used to interpret short-run dynamics and assess growth trends while removing regular seasonal fluctuations. For instance, the year-on-year change $\Delta_{12} X_t$ in monthly data smooths out within-year seasonality and is particularly useful for economic indicators such as industrial production, retail sales, or consumer prices.

4. Percent Change from Year Ago

A common way to measure the annual growth rate of a variable observed at high frequency (e.g. monthly or quarterly) is to compute the *percent change from a year ago*. A general measure of the annual (year-over-year) growth rate of a time series is

$$\% \Delta X_t^{(y)} = \frac{X_t - X_{t-s}}{X_{t-s}} \times 100,$$

where s denotes the number of observations per year (e.g. $s = 12$ for monthly data, $s = 4$ for quarterly data). The superscript (y) indicates that this is a *year-over-year* percentage change.

For quarterly data:

$$\% \Delta X_t^{(y)} = \frac{X_t - X_{t-4}}{X_{t-4}} \times 100,$$

since there are 4 quarters in a year. This measures how much the current quarter's value has increased or decreased relative to the same quarter one year earlier.

For monthly data:

$$\% \Delta X_t^{(y)} = \frac{X_t - X_{t-12}}{X_{t-12}} \times 100,$$

since there are 12 months in a year. This captures the percentage change of the variable compared to its value in the same month of the previous year.

Both transformations express how much X_t has changed compared to its value in the same period one year earlier, effectively controlling for seasonal patterns. It is widely used for reporting annual growth rates in macroeconomic indicators such as GDP, industrial production, or consumer prices, as they provide a smoother and more interpretable measure of underlying economic trends than short-term monthly or quarterly changes.

5. Compounded Annual Rate of Change (CARC)

The *Compounded Annual Rate of Change (CARC)* converts a short-term growth rate (e.g. quarterly or monthly) into an annualized rate, assuming that the same short-term rate were to continue for a full year.

$$\text{CARC}_t = \left[\left(\frac{X_t}{X_{t-1}} \right)^s - 1 \right] \times 100,$$

where s is the number of periods per year ($s = 4$ for quarterly data, $s = 12$ for monthly data).

Quarterly data:

$$\text{CARC}_t = \left[\left(\frac{X_t}{X_{t-1}} \right)^4 - 1 \right] \times 100$$

Monthly data:

$$\text{CARC}_t = \left[\left(\frac{X_t}{X_{t-1}} \right)^{12} - 1 \right] \times 100$$

Interpretation. CARC answers the question: “*If the most recent period-to-period growth rate continued for a full year, what would the implied annual rate of change be?*” It therefore *annualizes* the most recent growth rate, rather than measuring the actual change over the past year.

Comparison with year-over-year percentage change.

	Percent Change from Year Ago	Compounded Annual Rate of Change (CARC)
Formula	$\frac{X_t - X_{t-s}}{X_{t-s}} \times 100$	$\left[\left(\frac{X_t}{X_{t-1}} \right)^s - 1 \right] \times 100$
What it measures	Actual change compared with the same period one year earlier	Annualized version of the most recent short-term (monthly or quarterly) growth rate
Uses	Describes realized year-over-year growth (e.g. “GDP grew 2.1% compared with a year ago.”)	Shows implied annual rate if the current pace continued (e.g. “Annualized growth rate 3.5%.”)
Sensitivity	Reflects cumulative evolution over the past year	More sensitive to short-term fluctuations

Example. Suppose quarterly GDP increases from 100 to 101 this quarter. The quarter-on-quarter change is 1%:

$$\frac{101}{100} - 1 = 0.01.$$

If this quarterly growth persisted for four quarters, the annualized (compounded) rate would be

$$(1.01)^4 - 1 = 0.0406,$$

that is, approximately 4.1%. In contrast, the year-over-year change compares the current quarter’s level to the same quarter last year, which might be quite different (for example, 2.5% in the case: 100, 101, 101, 102, 102.5).

6. Continuously Compounded Rate of Change (CCR)

The *Continuously Compounded Rate of Change (CCR)* expresses the percentage change between two periods using natural logarithms:

$$\text{CCR}_t = (\ln X_t - \ln X_{t-1}) \times 100.$$

This measure can be interpreted as the growth rate that, if compounded continuously, would generate the same proportional change between X_{t-1} and X_t . It is sometimes called the *log return* or *instantaneous growth rate*.

Why use logs? For small changes, the log difference approximates the percentage change:

$$\ln X_t - \ln X_{t-1} \approx \frac{X_t - X_{t-1}}{X_{t-1}}.$$

Thus, CCR_t and the simple percent change are nearly identical when growth rates are moderate (say, below 10%). However, using logs has several advantages:

- Log changes are *time additive*: the sum of log changes over several periods equals the total log change.
- They handle compounding naturally.
- They symmetrically treat increases and decreases.

7. Continuously Compounded Annual Rate of Change (CCARC)

To express the continuously compounded rate on an *annualized* basis, multiply the per-period log change by the number of periods per year (s).

Quarterly data:

$$\text{CCARC}_t = (\ln X_t - \ln X_{t-1}) \times 400,$$

since there are 4 quarters in a year ($4 \times 100 = 400$).

Monthly data:

$$\text{CCARC}_t = (\ln X_t - \ln X_{t-1}) \times 1200,$$

since there are 12 months in a year ($12 \times 100 = 1200$).

Interpretation. The CCARC is the *annualized continuously compounded growth rate* implied by the most recent period's log change. It plays a similar role to the CARC defined earlier, but in log terms.

Comparison with other growth measures.

- The CCR and CCARC are very close to simple percentage changes when growth rates are small.
- Using logs makes growth rates additive across time, which simplifies analysis of long-term trends and model estimation.
- In empirical macroeconomics and finance, log differences (i.e. CCRs) are often preferred for stationarity and interpretability.

For example, a quarterly log growth of 0.00995 implies an annualized continuously compounded rate of

$$0.00995 \times 4 \times 100 = 3.98\%,$$

which corresponds closely to a 4% annualized simple rate.

8. Natural Log

Transformations such as logarithms

$$Y_t = \ln(X_t)$$

can help to stabilize the variance of a time series ... and many more. For a start see [this site](#) (symmetry, additivity but check out economics below - elasticity and semi-elasticity ...)

The linear relationship is given by:

$$Y = \alpha + \beta X$$

Interpretation:

- If $\beta > 0$ (positive relationship): an increase of one (1) unit in X leads to an **increase** of β units in Y .
- If $\beta < 0$ (negative relationship): an increase of one (1) unit in X leads to a **decrease** of β units in Y .

The log-log (double-log, elasticity) model is given by:

$$\ln(Y) = \alpha + \beta \ln(X)$$

Interpretation:

- If $\beta > 0$ (positive relationship): a 1% increase in X leads to an approximate $\beta\%$ **increase** in Y .
- If $\beta < 0$ (negative relationship): a 1% increase in X leads to an approximate $\beta\%$ **decrease** in Y .

The semi-log (semi-elasticity...) model is given by:

$$\ln(Y) = \alpha + \beta X$$

Interpretation:

- If $\beta > 0$ (positive relationship): an increase of one (1) unit in X leads to an approximate $100 \times \beta\%$ **increase** in Y .
- If $\beta < 0$ (negative relationship): an increase of one (1) unit in X leads to an approximate $100 \times \beta\%$ **decrease** in Y .

The semi-log model is given by:

$$Y = \alpha + \beta \ln(X)$$

Interpretation:

- If $\beta > 0$ (positive relationship): a 1% increase in X leads to an approximate **increase** of $\beta/100$ units in Y .
- If $\beta < 0$ (negative relationship): a 1% increase in X leads to an approximate **decrease** of $\beta/100$ units in Y .

Check out:

- [Understanding the Data](#)
- [Frequently Asked Questions](#)
 - [FRED – Recession Bars](#). Here, you will realize that you must visit [NBER](#). **Question:** What is NBER? **Answer:** The National Bureau of Economic Research (NBER) is a private, nonprofit organization that facilitates cutting-edge research on and analysis of major economic issues. More in class...

References

[FRED – Formulas to Calculate Growth Rates](#)