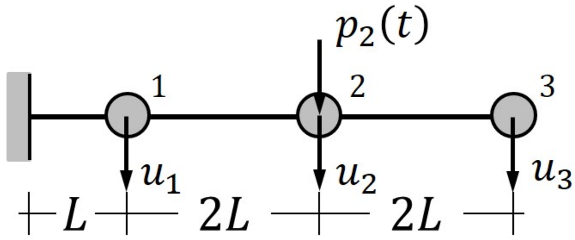
Η εικονιζομένη πρόβολος δοκός διακριτοποιείται με την χρήση τριών ίσων συγκεντρωμένων μαζών (εκάστη μαζα είναι ίση με $m = 5837\ kg$), όπως φαίνεται στο σχήμα. Τα δυναμικά χαρακτηριστικά της κατασκευής (ιδιομορφές $\boldsymbol{\phi}_n$ & κυκλικές ιδιοσυχνότητες ω_n: $n = 1,2,3$) δίδονται στους ακολούθους δύο πίνακες:



$$\boldsymbol{\Phi} = \begin{bmatrix} 0.054 & 0.283 & 0.957 \\ 0.406 & 0.870 & -0.281 \\ 0.913 & -0.402 & 0.068 \end{bmatrix} \quad , \quad \boldsymbol{\Omega} = \begin{bmatrix} 3.61 & & \\ & 24.2 & \\ & & 77.7 \end{bmatrix} (rad/sec)$$

Θεωρούμε ότι η δοκός έχει μηδενική απόσβεση.

(1) Υπολογίσατε ποία πρέπει να είναι η αρχική μετατόπιση
$$\mathbf{u}^T(t=0) = [u_1(t=0), u_2(t=0), u_3(t=0)]^T$$
της δοκού ετσι ώστε να διεγερθεί μόνον η $\boldsymbol{\phi}_3$, εάν $u_2(t=0) = 5\ cm$.

(2) Υπολογίσατε την απόκριση $u_3(t)$ της μάζας #3 όταν μια δύναμις $p_2(t) = 1 \cdot \delta(t)\ N$ ασκείται στην μάζα #2, υποθετοντας ότι η κατασκευη ευρισκεται σε κατάσταση ηρεμίας.

(3) Κάνοντας χρήση του αποτελέσματος του ερωτήματος (2) υπολογίσατε την απόκριση $u_3(t)$ της δοκού όταν
$$p_2(t) \quad = \quad \begin{cases} p_o \cdot \sin(\Omega t) & 0 \le t \le \left(\frac{\pi}{\Omega}\right) \\ 0 & \left(\frac{\pi}{\Omega}\right) < t \end{cases} \quad , \quad \left(\Omega = \frac{3}{4}\omega_1 \quad p_o = 10kN\right)$$
Υποθέτουμε ότι η κατασκευη εκκινεί από την κατάσταση ηρεμίας.

(4) Για την φόρτιση που διδεται στο ερωτημα (3), εκτιμήσατε την μεγιστη $\max_t |u_3(t)|$ απόκριση υποθετοντας ότι η ολικη απόκριση περιγράφεται ικανοποιητικά λαμβάνοντας υποψη την συμμετοχή μόνον της πρώτης ιδιομορφής.

(5) Υπολογίσατε την απόκριση $u_3(t)$, σε μόνιμη ταλάντωση (steady-state), της δοκού όταν η διεγείρουσα δύναμη, η ασκουμενη στην μαζα #2, είναι αρμονική της μορφής $p_2(t) = p_o \cdot e^{i\Omega t}$ ($\Omega = \frac{3}{4}\omega_1 \quad p_o = 10\ kN$). Εκτιμήσατε την μεγίστη απόκριση $\max_t |u_3(t)|$ υποθετοντας ότι η ολικη απόκριση περιγράφεται ικανοποιητικά λαμβάνοντας υποψη την συμμετοχή μόνον της πρώτης ιδιομορφής.

SOLUTION

In order to excite the 3rd mode only with the requirement that $u_2(t=0) = 5\ cm$, then $\mathbf{u}(0) = c\boldsymbol{\phi}_3$. Specifically,

$$\mathbf{u}(0) = \begin{Bmatrix} u_1(t=0) \\ u_2(t=0) \\ u_3(t=0) \end{Bmatrix} = c \begin{Bmatrix} 0.957 \\ -0.281 \\ 0.068 \end{Bmatrix} = \begin{Bmatrix} u_1(t=0) \\ 5 \\ u_3(t=0) \end{Bmatrix} cm$$

Evidently, $c = -17.794$. Therefore

$$\mathbf{u}(0) = \begin{Bmatrix} u_1(t=0) \\ u_2(t=0) \\ u_3(t=0) \end{Bmatrix} = c \begin{Bmatrix} 0.957 \\ -0.281 \\ 0.068 \end{Bmatrix} = -17.794 \begin{Bmatrix} 0.957 \\ -0.281 \\ 0.068 \end{Bmatrix} = \boxed{\begin{Bmatrix} -17.029 \\ 5.000 \\ -1{,}210 \end{Bmatrix} cm}$$

The applied force is $\mathbf{p}(t) = \mathbf{s}\delta(t)\ N$, where $\mathbf{s}^T = [0,1,0]^T$. We resolve vector $\mathbf{s}$ in its modal components:

$$\mathbf{s} = \sum_{n=1}^{N} \Gamma_n \mathbf{m}\boldsymbol{\phi}_n \qquad \Gamma_n = \frac{\boldsymbol{\phi}_n^T \mathbf{s}}{\boldsymbol{\phi}_n^T \mathbf{m}\boldsymbol{\phi}_n}$$

$$M_1 = \boldsymbol{\phi}_1^T \mathbf{m}\boldsymbol{\phi}_1 = 5844.71 \qquad M_2 = \boldsymbol{\phi}_2^T \mathbf{m}\boldsymbol{\phi}_2 = 5828.79 \qquad M_3 = \boldsymbol{\phi}_3^T \mathbf{m}\boldsymbol{\phi}_3 = 5833.7$$

$$\Gamma_1 = \frac{\boldsymbol{\phi}_1^T \mathbf{s}}{\boldsymbol{\phi}_1^T \mathbf{m}\boldsymbol{\phi}_1} = 6.946 \cdot 10^{-5} \quad \Gamma_2 = \frac{\boldsymbol{\phi}_2^T \mathbf{s}}{\boldsymbol{\phi}_2^T \mathbf{m}\boldsymbol{\phi}_2} = 1.493 \cdot 10^{-4} \quad \Gamma_3 = \frac{\boldsymbol{\phi}_3^T \mathbf{s}}{\boldsymbol{\phi}_3^T \mathbf{m}\boldsymbol{\phi}_3} = -4.817 \cdot 10^{-5}$$

$$\mathbf{m} = m \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 5837 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} kg$$

The displacement response to the force $\mathbf{p}(t) = \mathbf{s}\delta(t)$ is

$$\mathbf{u}(t) = \sum_{n=1}^{N} q_n(t)\boldsymbol{\phi}_n = \sum_{n=1}^{N} \Gamma_n D_n(t)\boldsymbol{\phi}_n$$

The modal equations and their solutions are

$$\left.\begin{matrix} \ddot{D}_n + \omega_n^2 D_n = \delta(t) \\ \begin{Bmatrix} D_1(t=0) = 0 \\ \dot{D}_2(t=0) = 0 \end{Bmatrix} \end{matrix}\right\} \Rightarrow D_n(t) = \frac{1}{\omega_n}\sin(\omega_n t) \quad (n = 1,2,3)$$

Therefore

$$\boxed{\begin{aligned} u_3(t) &= \Gamma_1 D_1(t)\phi_{31} + \Gamma_2 D_2(t)\phi_{32} + \Gamma_3 D_3(t)\phi_{33} \\ &= (6.946 \cdot 10^{-5})D_1(t) + (-4.194 \cdot 10^{-5})D_2(t) + (-3.275 \cdot 10^{-6})D_3(t) \end{aligned}}$$

When $p_2(t)$ is equal to the given half-sine pulse, the response $u_3(t)$ is

$$\begin{aligned} u_3(t) &= \Gamma_1[D_1(t) * p_2(t)]\phi_{31} + \Gamma_2[D_2(t) * p_2(t)]\phi_{32} + \Gamma_3[D_3(t) * p_2(t)]\phi_{33} \\ &= \left\{ \begin{matrix} (6.648 \cdot 10^{-5})[D_1(t) * p_2(t)] \\ + \\ (-4.194 \cdot 10^{-5})[D_2(t) * p_2(t)] \\ + \\ (-3.275 \cdot 10^{-6})[D_3(t) * p_2(t)] \end{matrix} \right\} \end{aligned}$$

The convolution integral $D_n(t) * p_n(t)$ is

$$D_n(t) * p_n(t) = \begin{cases} p_o \cdot \int_0^t \sin(\Omega\tau) \cdot \frac{1}{\omega_n} \sin[\omega_n(t-\tau)]\ d\tau & \left(t \leq \left(\frac{\pi}{\Omega}\right)\right) \\ p_o \cdot \int_0^{t_d=(\pi/\Omega)} \sin(\Omega\tau) \cdot \frac{1}{\omega_n} \sin[\omega_n(t-\tau)]\ d\tau & \left(\left(\frac{\pi}{\Omega}\right) < t\right) \end{cases}$$

Recall that: $\sin x \cdot \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$ and $\Omega = \frac{3}{4}\omega_1$; let $\beta_n = \frac{\Omega}{\omega_n}$; then

$$\begin{aligned} \sin(\Omega\tau) \cdot \sin[\omega_n(t-\tau)] &= \sin(\beta_n \omega_n \tau) \cdot \sin[\omega_n(t-\tau)] \\ &= \frac{1}{2}\left\{\cos\left[\omega_n\left((\beta_n+1)\tau - t\right)\right] - \cos\left[\omega_n\left((\beta_n-1)\tau + t\right)\right]\right\} \end{aligned}$$

Therefore

$$\int_0^t \sin(\Omega\tau)\cdot\frac{1}{\omega_n}\sin[\omega_n(t-\tau)]\ d\tau = \frac{1}{2\omega_n}\left\{\begin{array}{c}\displaystyle\int_0^t \cos\big[\omega_n\big((\beta_n+1)\tau-t\big)\big]\ d\tau \\ + \\ \displaystyle-\int_0^t \cos\big[\omega_n\big((\beta_n-1)\tau+t\big)\big]\ d\tau\end{array}\right.$$

$$= \frac{1}{2\omega_n}\left\{\begin{array}{c}\displaystyle\frac{1}{\omega_n(\beta_n+1)}\sin[\zeta]\Big|_{-\omega_n t}^{\omega_n\beta_n t} \\ + \\ \displaystyle\frac{-1}{\omega_n(\beta_n-1)}\sin[\zeta]\Big|_{\omega_n t}^{\omega_n\beta_n t}\end{array}\right.$$

$$= \frac{1}{2\omega_n^2}\left\{\begin{array}{c}\displaystyle\frac{1}{(\beta_n+1)}\{\sin(\Omega t)+\sin(\omega_n t)\} \\ + \\ \displaystyle\frac{-1}{(\beta_n-1)}\{\sin(\Omega t)-\sin(\omega_n t)\}\end{array}\right.$$

$$= \frac{1}{2\omega_n^2}\left[\frac{-2}{\beta_n^2-1}\sin(\Omega t)+\frac{2\beta_n}{\beta_n^2-1}\sin(\omega_n t)\right]$$

$$= \frac{1}{\omega_n^2\big(\beta_n^2-1\big)}[\beta_n\sin(\omega_n t)-\sin(\Omega t)]$$

$$= \frac{1}{\omega_n^2\big(1-\beta_n^2\big)}[\sin(\Omega t)-\beta_n\sin(\omega_n t)]$$

This result is identical (with the proper adjustments ($\times\,\Gamma_n$) with Eqn. 4.8.2 in Chopra's book.

Similarly

$$\int_0^{t_d=(\pi/\Omega)} \sin(\Omega\tau)\cdot\frac{1}{\omega_n}\sin[\omega_n(t-\tau)]\,d\tau = \frac{1}{2\omega_n}\begin{cases}\displaystyle\int_0^{t_d=(\pi/\Omega)} \cos\big[\omega_n\big((\beta_n+1)\tau-t\big)\big]\,d\tau \\ + \\ \displaystyle -\int_0^{t_d=(\pi/\Omega)} \cos\big[\omega_n\big((\beta_n-1)\tau+t\big)\big]\,d\tau\end{cases}$$

$$= \frac{1}{2\omega_n}\begin{cases}\displaystyle\frac{1}{\omega_n(\beta_n+1)}\sin[\zeta]\Big|_{-\omega_n t}^{\omega_n((\beta_n+1)t_d-t)} \\ + \\ \displaystyle\frac{-1}{\omega_n(\beta_n-1)}\sin[\zeta]\Big|_{\omega_n t}^{\omega_n((\beta_n-1)t_d+t)}\end{cases}$$

$$= \frac{1}{2\omega_n^2}\begin{cases}\displaystyle\frac{1}{(\beta_n+1)}\left\{\sin\left(\frac{\beta_n+1}{\beta_n}\pi-\omega_n t\right)+\sin(\omega_n t)\right\} \\ + \\ \displaystyle\frac{-1}{(\beta_n-1)}\left\{\sin\left(\frac{\beta_n-1}{\beta_n}\pi+\omega_n t\right)-\sin(\omega_n t)\right\}\end{cases}$$

$$= \frac{1}{2\omega_n^2}\begin{cases}\displaystyle\frac{1}{(\beta_n+1)}\left\{\sin(\omega_n t)+\sin\left(\omega_n t-\frac{\pi}{\beta_n}\right)\right\} \\ + \\ \displaystyle\frac{1}{(\beta_n-1)}\left\{\sin(\omega_n t)+\sin\left(\omega_n t-\frac{\pi}{\beta_n}\right)\right\}\end{cases}$$

$$= \frac{1}{2\omega_n^2(\beta_n^2-1)}\begin{cases}\displaystyle(\beta_n-1)\left\{\sin(\omega_n t)+\sin\left(\omega_n t-\frac{\pi}{\beta_n}\right)\right\} \\ + \\ \displaystyle(\beta_n+1)\left\{\sin(\omega_n t)+\sin\left(\omega_n t-\frac{\pi}{\beta_n}\right)\right\}\end{cases}$$

$$= \frac{\beta_n}{\omega_n^2(\beta_n^2-1)}\left\{\sin(\omega_n t)+\sin\left(\omega_n t-\frac{\pi}{\beta_n}\right)\right\}$$

$$= \frac{2\beta_n}{\omega_n^2(\beta_n^2-1)}\sin\left(\omega_n t-\frac{\pi}{2\beta_n}\right)\cos\left(\frac{\pi}{2\beta_n}\right)$$

$$= \frac{2\beta_n\cos\left(\frac{\pi}{2\beta_n}\right)}{\omega_n^2(\beta_n^2-1)}\sin\left(\omega_n t-\frac{\pi}{2\beta_n}\right)$$

This result is identical (with the proper adjustments ($\times\,\Gamma_n$) with Eqn. 4.8.3 in Chopra's book.

Summarizing:

$$D_n(t) * p_n(t) = \begin{cases} \dfrac{p_o}{\omega_n^2(1-\beta_n^2)}[\sin(\Omega t) - \beta_n \sin(\omega_n t)] & \left(t \le \left(\dfrac{\pi}{\Omega}\right)\right) \\ \dfrac{2\beta_n \cos\left(\dfrac{\pi}{2\beta_n}\right) \cdot p_o}{\omega_n^2(\beta_n^2 - 1)} \sin\left(\omega_n t - \dfrac{\pi}{2\beta_n}\right) & \left(\left(\dfrac{\pi}{\Omega}\right) < t\right) \end{cases}$$

Notice that in the forced vibration part $\left(t \le (\pi/\Omega)\right)$ both the forcing circular frequency Ω and the natural circular frequency ω_n of the corresponding mode are involved in the oscillatory response. For the free-vibration part $\left((\pi/\Omega) < t\right)$ only the natural circular frequency ω_n of the corresponding mode are involved in the oscillatory response.

Determination of $\max_t |u_3(t)|$, assuming that the total response is satisfactorily described by the first mode alone.

Evidently

$$\frac{t_d}{T_1} = \frac{\left(\frac{\pi}{\Omega}\right)}{T_1} = \frac{\omega_1}{2\Omega} = \frac{\omega_1}{2\left(\frac{3}{4}\omega_1\right)} = \frac{2}{3}$$

Because, $0.5 \le (t_d/T_1) \le 1.5$, the peak is controlled by the forced phase, during which only one peak occurs (see Chopra, Section 4.8). The peak is given by Eqn. 4.8.9, properly interpreted / modified for our problem. Specifically

$$\max_t |u_3(t)| = \frac{\Gamma_1 \cdot \phi_{31} \cdot p_o}{\omega_1^2\left(1-\beta_1^2\right)}\left(\sin\left(\frac{2\pi\beta_1}{1+\beta_1}\right) - \beta_1 \sin\left(\frac{2\pi}{1+\beta_1}\right)\right)$$

For our problem

$$\omega_1 = 3.61 \frac{rad}{sec} \qquad \beta_1 = \frac{\Omega}{\omega_1} = \frac{\frac{3}{4}\omega_1}{\omega_1} = \frac{3}{4}$$

Therefore

$$\max_t |u_3(t)| = (8.853 \cdot 10^{-6} \cdot 10^4)\, m = 8.853\ cm$$

Steady state response:

The applied force is $\mathbf{p}(t) = \mathbf{s}e^{i\Omega t}$, where $\mathbf{s}^T = [0,1,0]^T$, i.e. the vector s which defines the spatial distribution of the applied force is the same as in question (2).

$$[\mathbf{u}(t)]_{ss} = \sum_{n=1}^{N} [q_n(t)]_{ss} \boldsymbol{\Phi}_n = \sum_{n=1}^{N} \Gamma_n [D_n(t)]_{ss} \boldsymbol{\Phi}_n$$

The modal equations are

$$\ddot{D}_n + \omega_n^2 D_n = p_o \cdot e^{i\Omega t} \quad (n = 1,2,3)$$

The steady-state response is $[D_n(t)]_{ss} = H_n(\Omega) \cdot e^{i\Omega t}$. Substituting in the ODE we convert it to an algebraic equation. Specifically

$$\ddot{D}_n + \omega_n^2 D_n = p_o \cdot e^{i\Omega t} \quad \Rightarrow \quad [(i\Omega)^2 H_n(\Omega) + \omega_n^2 H_n(\Omega)] \cdot e^{i\Omega t} = p_o \cdot e^{i\Omega t} \quad \Rightarrow \quad H_n(\Omega) = \frac{p_o}{\omega_n^2 - \Omega^2}$$

$$\boxed{[u_3(t)]_{ss} = \sum_{n=1}^{3} \frac{\Gamma_n \phi_{3n} p_o}{\omega_n^2 - \Omega^2} e^{i\Omega t}}$$

Peak response, assuming that the total response is satisfactorily described by the first mode alone:

$$\boxed{|[u_3(t)]_{ss}|_o = \frac{\Gamma_1 \phi_{31} p_o}{\omega_1^2 - \Omega^2} = \frac{\Gamma_1 \phi_{31} p_o}{\omega_1^2 \left[1 - \left(\frac{3}{4}\right)^2\right]} = (1.166 \cdot 10^{-5} \cdot 10^4)\, m = 11.7\, cm}$$