MODAL RESPONSE CONTRIBUTIONS

Modal Expansion of Excitation Vector $\mathbf{p}(t) = \mathbf{s}p(t)$

A common loading case is the one in which the applied forces $p_j(t)$ have the **same time variation** $p(t)$, and their **spatial distribution** is defined by $\mathbf{s}$, **independent of time**.

$$\mathbf{p}(t) = \mathbf{s}p(t)$$

We **expand** the vector $\mathbf{s}$ in its modal components as:

$$\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m} \boldsymbol{\phi}_r$$

Pre-multiplying by $\boldsymbol{\phi}_n^T$ and utilizing the orthogonality property of the modes, we obtain:

$$\Gamma_n = \frac{\boldsymbol{\phi}_n^T \mathbf{s}}{M_n} \qquad \textbf{Modal Participation Factor}$$

It is evident that Γ_n is **not independent of how the mode is normalized**.

The contribution of the n^{th} mode to $\mathbf{s}$ is:

$$\mathbf{s}_n = \Gamma_n \mathbf{m} \boldsymbol{\phi}_n$$

which **is independent of how the modes are normalized**.

The inertia forces, associated with the n^{th} mode, are:

$$(\mathbf{f}_I)_n = -\mathbf{m}\ddot{\mathbf{u}}_n(t) = -\mathbf{m}\boldsymbol{\phi}_n \ddot{q}_n(t)$$

Clearly, **the spatial distribution of the inertia forces $(\mathbf{f}_I)_n$ associated with the n^{th} mode, is the same as that of $\mathbf{s}_n$.**

The expansion,

$$\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m} \boldsymbol{\phi}_r$$

has two useful properties:

(1) The force vector $\mathbf{s}_n p(t)$ produces response only in the n^{th} mode but <u>no response in any other mode</u>.

<u>Proof</u>:

Generalized force associated with $\mathbf{s}_n p(t)$:

$$P_r(t) = \boldsymbol{\phi}_r^T \mathbf{s}_n p(t) = \Gamma_r \underbrace{(\boldsymbol{\phi}_r^T \mathbf{m} \boldsymbol{\phi}_n)}_{M_n \delta_{nr}} p(t) = \Gamma_r M_n \delta_{nr} p(t)$$

$$[\delta_{nr}: Kronecker\ delta]$$

(2) The dynamic response of the nth mode is entirely due to the partial force vector $\mathbf{s}_n p(t)$.

<u>Proof</u>:

Generalized force associated with $\mathbf{s} p(t)$:

$$P_n(t) = \boldsymbol{\phi}_n^T \mathbf{s} p(t) = \sum_{r=1}^{N} \Gamma_r (\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_r) p(t) = \Gamma_n M_n p(t)$$

Evidently, the n^{th} mode generalized force associated with the complete force vector $\mathbf{s} p(t)$ is the same as the n^{th} mode generalized force associated with the partial force vectors $\mathbf{s}_n p(t)$.

Modal Analysis for $\boxed{\mathbf{p}(t) = \mathbf{s}p(t)}$

We have derived N **uncoupled equations**:

$$\ddot{q}_n(t) + 2\xi_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = \frac{P_n(t)}{M_n} \quad (n = 1,2,\cdots,N)$$

$$\text{where:} \quad P_n(t) = \boldsymbol{\phi}_n^T\mathbf{p}(t) = \boldsymbol{\phi}_n^T\mathbf{s}p(t) = \Gamma_n M_n p(t)$$

Therefore:

$$\ddot{q}_n(t) + 2\xi_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = \Gamma_n p(t)$$

We can write the solution $q_n(t)$ as follows:

$$\boxed{\begin{gathered}\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = p(t) \\ \text{and} \quad q_n(t) = \Gamma_n D_n(t)\end{gathered}}$$

Contribution of the n^{th} mode to $\mathbf{u}(t)$:

$$\boxed{\mathbf{u}_n(t) = \boldsymbol{\phi}_n q_n(t) = \Gamma_n\boldsymbol{\phi}_n D_n(t)}$$

Equivalent static forces $\mathbf{f}_{Sn}(t)$:

$$\mathbf{f}_{Sn}(t) = \mathbf{k}\mathbf{u}_n(t) = \Gamma_n \underbrace{\mathbf{k}\boldsymbol{\phi}_n}_{\omega_n^2\mathbf{m}\boldsymbol{\phi}_n} D_n(t) = \underbrace{\Gamma_n\mathbf{m}\boldsymbol{\phi}_n}_{\mathbf{s}_n}[\omega_n^2 D_n(t)]$$

$$\Rightarrow \quad \boxed{\mathbf{f}_{Sn}(t) = \mathbf{s}_n[\omega_n^2 D_n(t)]}$$

Equivalent Static Forces $\mathbf{f}_{Sn}(t)$:

$$\boxed{\mathbf{f}_{Sn}(t) = \mathbf{s}_n[\omega_n^2 D_n(t)]}$$

The n^{th} mode contribution $r_n(t)$ to **any response quantity** $r(t)$ is **determined by static analysis of the structure subjected to forces** $\mathbf{f}_{Sn}(t)$.

Let:

r_n^{st} *modal static response* = **the static value of response quantity r due to external forces $\mathbf{s}_n$**

Then:

$$\boxed{r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_n^{st}[\omega_n^2 D_n(t)]}$$

The modal analysis procedure presented above **provides a basis for identifying and understanding the factors that influence the relative modal contributions to the response**.

NOTE:

Although we loosely refer to $\mathbf{s}_n$ as forces, **they are dimensionless** because $p(t)$ **has units of force**.

Thus, r_n^{st} **does not have the same units as** r, but equation $r_n(t) = r_n^{st}[\omega_n^2 D_n(t)]$ gives the correct units for r_n and equation $r(t) = \sum_{n=1}^{N} r_n(t)$ for r .

Modal Contribution Factors

Let:

r^{st} **= the static value of response quantity r due to external forces $\mathbf{s}$**

Contribution of the n^{th} mode to response quantity r :

$$r_n(t) = r_n^{st}[\omega_n^2 D_n(t)] = r^{st}\left(\frac{r_n^{st}}{r^{st}}\right)[\omega_n^2 D_n(t)]$$

$$= r^{st}\bar{r}_n[\omega_n^2 D_n(t)]$$

where: $$\boxed{\bar{r}_n = \left(\frac{r_n^{st}}{r^{st}}\right)}$$ ***n^{th} modal contribution factor***

Properties of modal contribution factors $\bar{r}_n$:

(1) They are **dimensionless;**

(2) They are **independent of how modes are normalized;**

(3) **The sum** of the modal contribution factors over all modes is unity, that is:

$$\boxed{\sum_{n=1}^{N} \bar{r}_n = 1}$$

Property #3 follows by the following reasoning:

$$\mathbf{s} = \sum_{n=1}^{N} \mathbf{s}_n \Rightarrow r^{st} = \sum_{n=1}^{N} r_n^{st} \Rightarrow 1 = \sum_{n=1}^{N} \underbrace{\left(\frac{r_n^{st}}{r^{st}}\right)}_{\bar{r}_n}$$

MODAL ANALYSIS: SUMMARY

Dynamic response of an MODF system with classical damping to external forces $\mathbf{p}(t)$:

(1) Define the **structural properties**:
   (a) Determine the **mass matrix** $\mathbf{m}$ and **stiffness matrix** $\mathbf{k}$;
   (b) Estimate the modal damping ratios ξ_n.

(2) **Determine the natural frequencies** ω_n and **modes** $\boldsymbol{\phi}_n$;
   [*i.e.*, solve the **eigenvalue problem**: $(\mathbf{k} - \omega^2 \mathbf{m})\boldsymbol{\phi} = \mathbf{0}$]

(3) **Compute the response in each mode** by following the steps:
   (a) Set up the following N **uncoupled equations** $(n = 1,2,\cdots,N)$:

$$M_n \ddot{q}_n(t) + C_n \dot{q}_n(t) + K_n q_n(t) = P_n(t)$$

$$\Longrightarrow \quad \ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = \frac{P_n(t)}{M_n}$$

   and solve for $q_n(t)$.

   (b) Compute the **nodal displacements** $\mathbf{u}_n(t) = \boldsymbol{\phi}_n q_n(t)$

   (c) Compute the **element forces** associated with $u_n(t)$ by implementing the ***equivalent static force method***:

$$\mathbf{f}_{Sn}(t) = \mathbf{k}\mathbf{u}_n(t) \quad \Longrightarrow \quad \mathbf{f}_{Sn}(t) = \omega_n^2 \mathbf{m} \boldsymbol{\phi}_n q_n(t) \quad (n = 1,2,\cdots,N)$$

(4) Combine the contributions of all the modes to determine the response:

e.g.

nodal displacements

$$\mathbf{u}(t) = \sum_{n=1}^{N} \mathbf{u}_n(t) = \sum_{n=1}^{N} \boldsymbol{\phi}_n q_n(t)$$

element forces

$$r(t) = \sum_{n=1}^{N} r_n(t)$$

MODAL ANALYSIS FOR $\mathbf{p}(t) = \mathbf{s}p(t)$

After we determine:

- the **structural properties** (i.e., $\mathbf{m}$ & $\mathbf{k}$, ξ_n).
- the **vibration properties** (i.e., ω_n & $\boldsymbol{\phi}_n$).

we expand the force distribution $\mathbf{s}$ into its modal components $\mathbf{s}_n$, *i.e.*,

$$\mathbf{s} = \sum_{n=1}^{N} \mathbf{s}_n = \sum_{n=1}^{N} \{\Gamma_n \mathbf{m} \boldsymbol{\phi}_n\}$$

The contribution $r_n(t)$ of the n^{th} mode to the dynamic response is obtained **by multiplying the results of two analyses**:

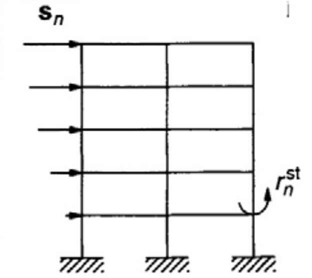
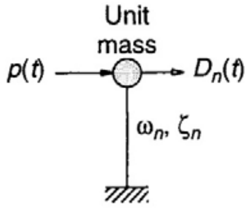
(1) ***Static analysis*** of the structure subjected to external forces $\mathbf{s}_n$ $(n = 1,2,\cdots,N)$.

(2) ***Dynamic analysis*** of the n^{th} mode SDOF system excited by the force $p(t)$:

$$\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = p(t) \quad (n = 1,2,\cdots,N)$$
$$q_n(t) = \Gamma_n D_n(t)$$

Combining the modal responses gives the dynamic response of the structure.

Conceptual explanation of modal analysis

<table>
<tr><th>Mode</th><th>Static Analysis of Structure</th><th>Dynamic Analysis of SDF System</th><th>Modal Contribution to Dynamic Response</th></tr>
<tr><td>n</td><td>
</td><td>
</td><td>$r_n(t) = r_n^{st}\,[\omega_n^2\, D_n(t)]$</td></tr>
</table>

Modal Responses: $n = 1, 2, \ldots N$

$$r_n(t) = r_n^{st}\,[\omega_n^2\, D_n(t)]$$

Total Response

$$r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_n^{st}\,[\omega_n^2\, D_n(t)]$$

Modal Responses and Required Number of Modes

Let:

$$D_{no} \stackrel{\text{def}}{=} \max_t |D_n(t)|$$

Then, the corresponding value of $r_n(t)$ is:

$$\boxed{r_{no} = \underbrace{r^{st}\bar{r}_n}_{r_n^{st}} \omega_n^2 D_{no}}$$

Let:

$$R_{dn} = \frac{D_{no}}{\left(D_{n,st}\right)_o} \quad \textit{Dynamic Response Factor}$$

where:

$$\begin{aligned}\left(D_{n,st}\right)_o &= \max_t \underbrace{\left|D_{n,st}(t)\right|}_{\textbf{static response}} \\ &= \left(\frac{\max_t |p(t)|}{\omega_n^2}\right) \\ &= \left(\frac{p_o}{\omega_n^2}\right)\end{aligned}$$

The ***static response*** $D_{n,st}(t)$ is obtained from $\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = p(t)$, **by dropping the** $\ddot{D}_n$ **&** $\dot{D}_n$ **terms**.

Therefore:

$$\boxed{r_{no} = r^{st}\bar{r}_n p_o R_{dn}}$$

The algebraic sign of r_{no} **is the same as that of** $r_n^{st} \stackrel{\text{def}}{=} r^{st}\bar{r}_n$ **because** R_{dn} **is positive by definition**.

Peak Modal Response

$$r_{no} = r^{st}\bar{r}_n p_o R_{dn}$$

Peak modal response is the product of four quantities:

(1) R_{dn} (=**dimensionless**) ***dynamic response factor*** for the n^{th} mode SDOF excited by force $p(t)$

(2) p_o, the ***peak value*** of $p(t)$

(3) $\bar{r}_n$ (=**dimensionless**) ***modal contribution factor*** for response quantity r

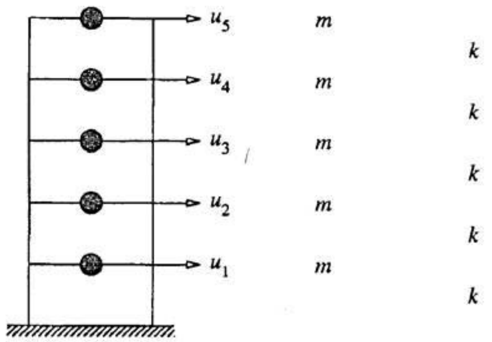
(4) r^{st}, **the** ***static value*** **of** r due to external forces $\mathbf{s}$

Clearly:

r^{st} & $\bar{r}_n$:	**depend on** the **spatial distribution** $\mathbf{s}$ of the applied forces but are **independent of** the **time variation** $p(t)$ of the applied forces.
R_{dn} :	**depends on** $p(t)$, but is **independent of** $\mathbf{s}$.

$\bar{r}_n$ & R_{dn} **influence the relative response contributions of the various vibration modes, and hence the minimum number of modes that should be included in the dynamic analysis.**

EXAMPLE:

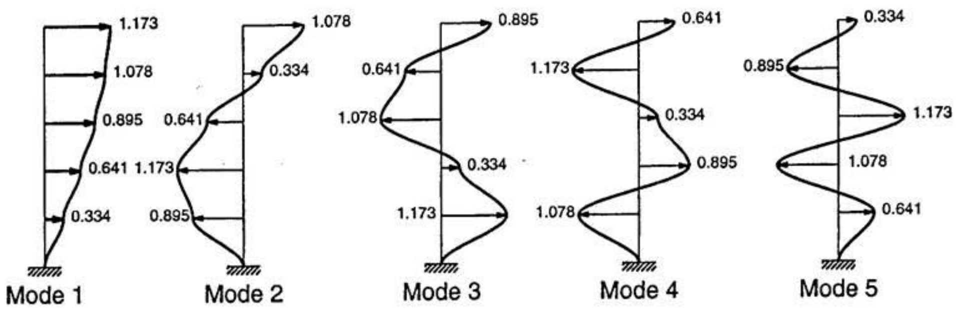


Uniform 5 – story shear building

The mass and stiffness matrices of the structure are

$$\mathbf{m} = m\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Natural vibration modes of five-story shear building



The **relative contributions** of the various modes to the response,

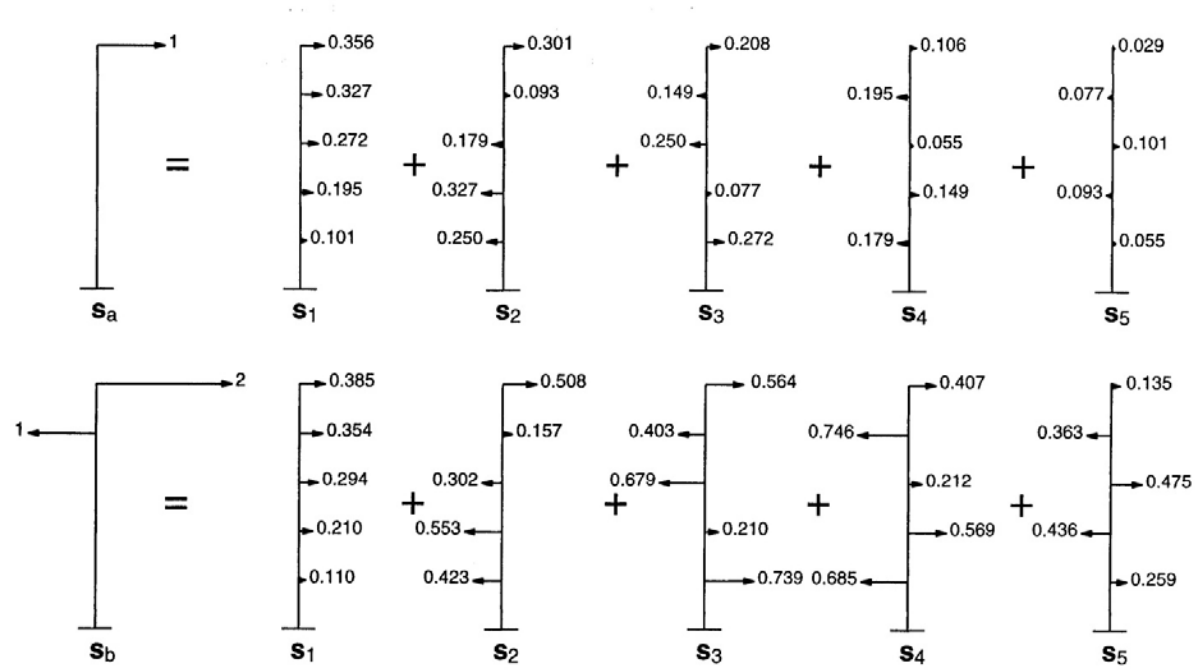
and

the **number of modes** that **should be included** in dynamic analysis

depend on the following **two influencing factors**:

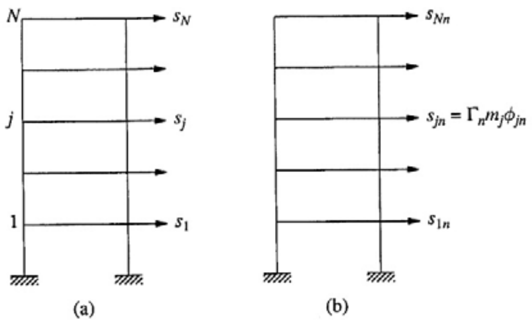
(1) The ***modal contribution factor*** $\bar{r}_n$, which depends on the **spatial distribution of forces**.
(2) The ***dynamic response factor*** R_{dn}, which is controlled by the **time variation of forces**.

Modal expansion of excitation vectors $\mathbf{s}_a$ and $\mathbf{s}_b$



<u>Observation</u>:

The contributions of the **<u>higher</u> modes** to $\mathbf{s}$ are **larger for** $\mathbf{s}_b$ than for $\mathbf{s}_a$, suggesting that these modes (*i.e.*, the higher modes) **may contribute more to the response if the force distribution is** $\mathbf{s}_b$ than if it is $\mathbf{s}_a$.



Determination of the ***modal contribution factors*** for:

(i) The base shear V_b
(ii) The roof (Nth floor) displacement u_N

$$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n \sum_{j=1}^{N} m_j \phi_{jn} =$$

$$\mathbf{u}_n^{st} = \mathbf{k}^{-1}\mathbf{s}_n = \mathbf{k}^{-1}\Gamma_n \mathbf{m}\boldsymbol{\phi}_n = \frac{\Gamma_n}{\omega_n^2}\boldsymbol{\phi}_n$$

(Recall that: $\mathbf{k}\boldsymbol{\phi}_n = \omega_n^2 \mathbf{m}\boldsymbol{\phi}_n \quad \Leftrightarrow \quad \frac{1}{\omega_n^2}\boldsymbol{\phi}_n = \mathbf{k}^{-1}\mathbf{m}\boldsymbol{\phi}_n$)

Therefore:

$$u_{Nn}^{st} = \frac{\Gamma_n}{\omega_n^2}\phi_{Nn} \qquad \begin{matrix} roof\ (N^{th}\ floor) \\ displacement \\ for\ n^{th}\ mode \end{matrix}$$

V_b^{st} & u_N^{st} are obtained **from a static analysis** of the building subjected **to (lateral) forces $\mathbf{s}$**.

We define:

$$\bar{V}_{bn} = \frac{V_{bn}^{st}}{V_b^{st}} \quad \& \quad \bar{u}_{Nn} = \frac{u_{Nn}^{st}}{u_N^{st}}$$

TABLE 12.11.1 MODAL AND CUMULATIVE CONTRIBUTION FACTORS

	Force Distribution, s_a				Force Distribution, s_b			
	Roof Displacement		Base Shear		Roof Displacement		Base Shear	
Mode n or Number of Modes, J	$\bar{u}_{5n}$	$\sum_{n=1}^{J} \bar{u}_{5n}$	$\bar{V}_{bn}$	$\sum_{n=1}^{J} \bar{V}_{bn}$	$\bar{u}_{5n}$	$\sum_{n=1}^{J} \bar{u}_{5n}$	$\bar{V}_{bn}$	$\sum_{n=1}^{J} \bar{V}_{bn}$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	−0.362	0.890	0.123	0.915	−0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.431	1.172
4	0.008	0.998	−0.063	0.985	0.024	0.994	−0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

Observations:

(1) The ***modal contribution factors*** are **larger for base shear than for roof displacement.**

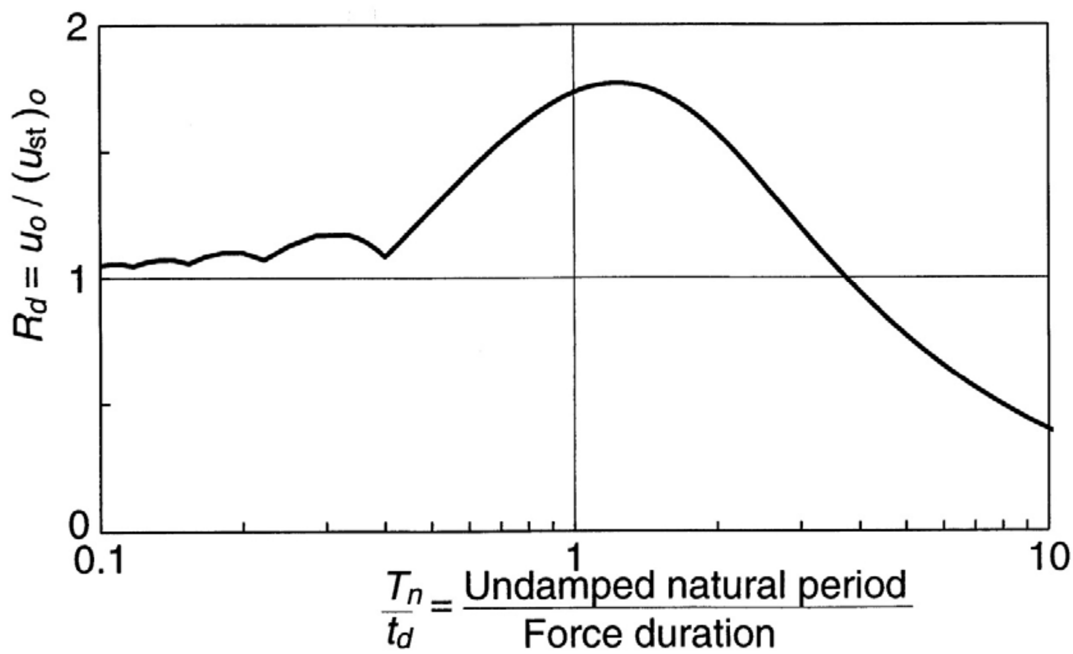
Higher modes contribute more to base shear (and other element forces) **than to roof displacement** (and other floor displacements)

(2) The ***modal contribution factors*** **for higher modes** are **larger** for force distribution $\mathbf{s}_b$ than for $\mathbf{s}_a$.

Higher modes contribute more to a response in the $\mathbf{s}_b$ case.

The modal expansion of $\mathbf{s}_a$ & $\mathbf{s}_b$ suggests the same conclusion.

Dynamic response factor for half-cycle sine pulse force



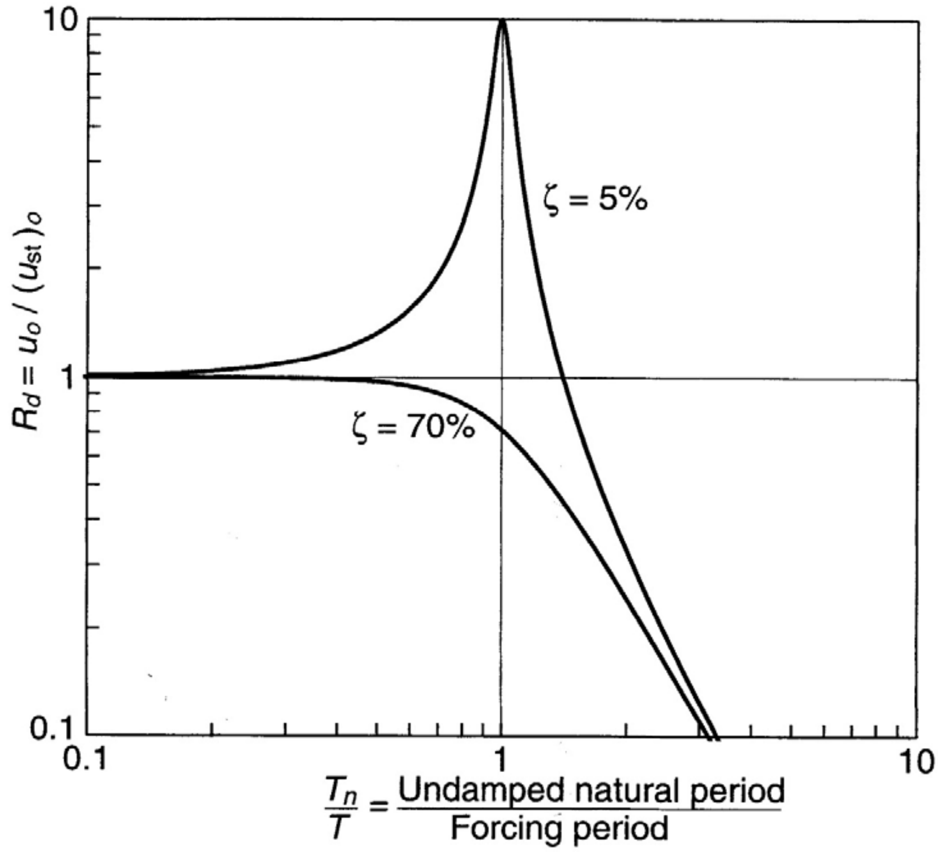
How R_{dn} (= the value of R_d for the n^{th} mode) **for a given excitation** $p(t)$ **varies with** n

depends on

where the natural periods T_n **fall on the period scale**.

In the case of **pulse excitation**, the above FIGURE shows that R_{dn} **varies over a narrow range** for a wide range of T_n and could have **similar values for several modes**.

Dynamic response factor for harmonic force



For **lightly damped systems** subjected **to harmonic excitation** (Ω), R_{dn} is especially large for modes with natural period T_n close to the **forcing period** T,

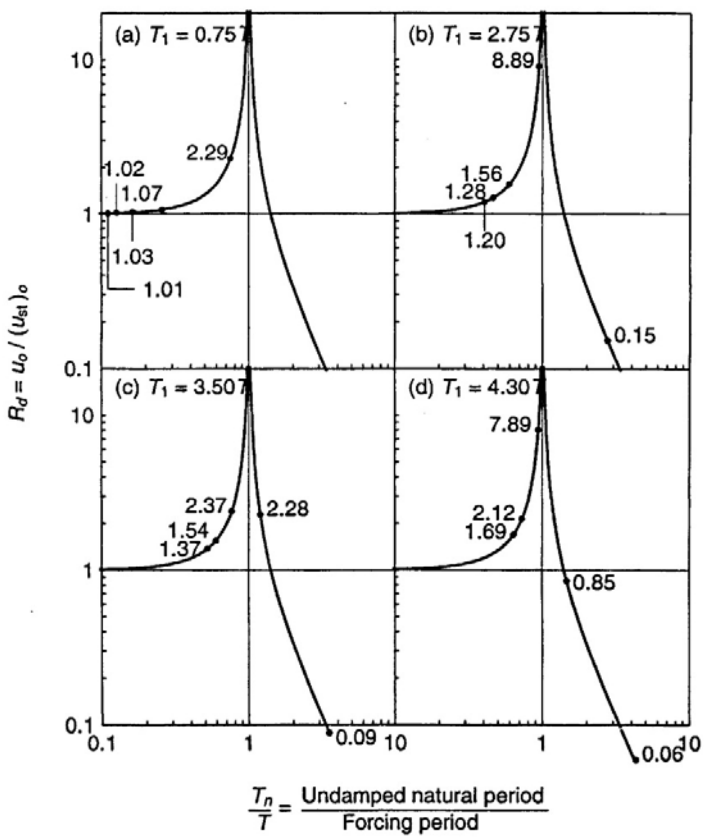
and **perhaps** these are the only modes that need be included in modal analysis

unless the modal contribution factors $\bar{r}_n$ for these modes are much smaller than for some other modes.

TABLE 12.11.1 MODAL AND CUMULATIVE CONTRIBUTION FACTORS

	Force Distribution, $\mathbf{s}_a$				Force Distribution, $\mathbf{s}_b$			
	Roof Displacement		Base Shear		Roof Displacement		Base Shear	
Mode n or Number of Modes, J	$\bar{u}_{5n}$	$\sum_{n=1}^{J} \bar{u}_{5n}$	$\bar{V}_{bn}$	$\sum_{n=1}^{J} \bar{V}_{bn}$	$\bar{u}_{5n}$	$\sum_{n=1}^{J} \bar{u}_{5n}$	$\bar{V}_{bn}$	$\sum_{n=1}^{J} \bar{V}_{bn}$
1	0.880	0.880	1.252	1.252	0.792	0.792	1.353	1.353
2	0.087	0.967	−0.362	0.890	0.123	0.915	−0.612	0.741
3	0.024	0.991	0.159	1.048	0.055	0.970	0.431	1.172
4	0.008	0.998	−0.063	0.985	0.024	0.994	−0.242	0.930
5	0.002	1.000	0.015	1.000	0.006	1.000	0.070	1.000

Dynamic response factors R_{dn}



In judging **the contribution of a natural mode** to the dynamic response of a structure, it is necessary to **consider the combined effects** of the modal contribution factor $\bar{r}_n$ and the **dynamic response factor** R_{dn}.