FREE VIBRATION RESPONSE OF UNDAMPED SYSTEMS

$$\begin{array}{|ll|} \hline \textit{Equation of Motion}: & \mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{0} \\ \textit{Initial Conditions}: & \mathbf{u}_0 = \mathbf{u}(0) \quad , \quad \dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0) \\ \hline \end{array}$$

As demonstrated previously, the above Equation of Motion (**free-vibration** equation) has a solution $\mathbf{u} = \boldsymbol{\phi}\rho\sin(\omega t + \theta)$ in which $\boldsymbol{\phi}$ and ω satisfy the ***Generalized Eigenvalue Problem***:

$$\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi} \tag{1}$$

where: $\lambda = \omega^2$

For an N -DOF system, equation (1) gives N values of λ , all of which are real and positive but may not be distinct.

Corresponding to each eigenvalue there is an associated eigenvector. These vectors are determined within a scalar multiple and form an orthogonal set.

In vibration analysis, the eigenvalues λ, or rather their square root ω, are referred to as the ***frequencies*** and the eigenvectors as the ***mode shapes*** (or ***normal modes***).

The **most general solution** of the free-vibration equation (***homogeneous*** **Equation of Motion**) is obtained as a **superposition of the** N **mode shapes** and can be written as:

$$\mathbf{u}(t) = \sum_{n=1}^{N} \rho_n \boldsymbol{\phi}_n \sin(\omega_n t + \theta_n)$$

$$or \quad \mathbf{u}(t) = \sum_{n=1}^{N} [A_n \boldsymbol{\phi}_n \cos(\omega_n t) + B_n \boldsymbol{\phi}_n \sin(\omega_n t)]$$

$$= \sum_{n=1}^{N} \underbrace{[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]}_{q_n(t)} \boldsymbol{\phi}_n$$

Where A_n & B_n (or equivalently ρ_n & θ_n) are arbitrary constants to be determined by the $2N$ **initial conditions** (recall that N **is the number of DOF's**).

Determination of the constants:

$$\dot{\mathbf{u}}(t) = \sum_{n=1}^{N} \underbrace{\omega_n[-A_n \sin(\omega_n t) + B_n \cos(\omega_n t)]}_{\dot{q}_n(t)} \boldsymbol{\phi}_n$$

Therefore:

$$\mathbf{u}(0) = \sum_{n=1}^{N} \underbrace{A_n}_{q_n(0)} \boldsymbol{\phi}_n \quad , \quad \dot{\mathbf{u}}(0) = \sum_{n=1}^{N} \underbrace{\omega_n B_n}_{\dot{q}_n(0)} \boldsymbol{\phi}_n$$

It follows that:

$$\boxed{q_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \mathbf{u}_0}{M_n} \quad , \quad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \dot{\mathbf{u}}_0}{M_n}}$$

Therefore, the ***free vibration response*** is:

$$\mathbf{u}(t) = \sum_{n=1}^{N} q_n(t) \boldsymbol{\phi}_n$$

where: $q_n(t) = q_n(0)\cos(\omega_n t) + \dfrac{\dot{q}_n(0)}{\omega_n}\sin(\omega_n t)$

FREE VIBRATION OF SYSTEMS WITH (NON-ZERO) DAMPING

$$\text{Equation of Motion:} \quad \mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = \mathbf{0}$$
$$\text{Initial Conditions:} \quad \mathbf{u}_0 = \mathbf{u}(0) \quad , \quad \dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0)$$

We are seeking a solution to the above problem. Procedures to obtain the desired solution differ depending on the form of damping: ***classical*** or ***non-classical***; these terms are defined next.

For **civil engineering structures** it is reasonable to assume that the matrices $\mathbf{m}$, $\mathbf{k}$, & $\mathbf{c}$ are all ***symmetric*** and ***positive definite***. [NOTE: For the proof to be developed below it suffices to have only $\mathbf{m}$ *positive definite* and the other two matrices, $\mathbf{k}$, & $\mathbf{c}$, *non-negative definite*.]

We are going to demonstrate that the ***necessary*** and ***sufficient*** condition for the above system to possess natural modes of vibration, that are **real-valued and identical to those of the associated undamped system** (also referred to as ***classical normal modes***), is the following equality:

$$\mathbf{cm^{-1}k} \quad = \quad \mathbf{km^{-1}c}$$

or, **equivalently**, that $(\mathbf{m^{-1}k})$ & $(\mathbf{m^{-1}c})$ commute.

The development is based on:

> **CAUGHEY, T. K., and O'KELLY, M. E. J., "Classical Normal Modes in Damped Linear Dynamic Systems," *Journal of Applied Mechanics*, ASME, 32, 1965, pp. 583–588.**

Proof:

Since $\mathbf{m}$ is **symmetric** and **positive definite**, it is always possible to find a ***congruence transformation*** ($[\mathbf{\Theta}]^T\mathbf{m}[\mathbf{\Theta}]$ where $[\mathbf{\Theta}]$ is **non-singular**, *e.g.*, HILDEBRAND, 1965, page 42; MEYER, 2000, page 568) which will reduce $\mathbf{m}$ to an identity matrix.

> **HILDEBRAND, F.B. (1965). *Methods of Applied Mathematics*, PRENTICE-HALL**
>
> **MEYER, C.D. (2000). *Matrix Analysis and Applied Linear Algebra*, SIAM**

NOTE: The reader should realize that in *module #11* we have established that the mass matrix may be written as $\mathbf{m} = \mathbf{Q}^T\mathbf{Q}$, where $\mathbf{Q}$ is **non-singular** and $(\mathbf{Q}^T)^{-1} = (\mathbf{Q}^{-1})^T$. Therefore $\mathbf{m} = \mathbf{Q}^T\mathbf{Q} \Leftrightarrow (\mathbf{Q}^T)^{-1}\mathbf{m}\mathbf{Q}^{-1} = \mathbf{I} \Leftrightarrow (\mathbf{Q}^{-1})^T\mathbf{m}\mathbf{Q}^{-1} = \mathbf{I}$. It must be evident to the reader that the **non-singular** matrix $[\mathbf{\Theta}]$ that we are seeking is $\mathbf{Q}^{-1}$ *i.e.*, $[\mathbf{\Theta}] \stackrel{\text{def}}{=} \mathbf{Q}^{-1}$.

Let this transformation be:

$$\mathbf{u}(t) \quad = \quad [\boldsymbol{\Theta}]\,\mathbf{z}(t)$$

So that:

$$[\boldsymbol{\Theta}]^T\mathbf{m}[\boldsymbol{\Theta}] \quad = \quad \mathbf{I}$$

Therefore:

$$[\boldsymbol{\Theta}]^T \cdot| \quad \left.\begin{array}{l}\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \\ \mathbf{u} = [\boldsymbol{\Theta}]\cdot\mathbf{z}\end{array}\right\} \quad \Rightarrow \quad \underbrace{[\boldsymbol{\Theta}]^T\mathbf{m}[\boldsymbol{\Theta}]}_{\mathbf{I}}\,\ddot{\mathbf{z}} + \underbrace{[\boldsymbol{\Theta}]^T\mathbf{c}[\boldsymbol{\Theta}]}_{\mathbf{A}}\,\dot{\mathbf{z}} + \underbrace{[\boldsymbol{\Theta}]^T\mathbf{k}[\boldsymbol{\Theta}]}_{\mathbf{B}}\,\mathbf{z} = \mathbf{0}$$

Therefore:

$$\mathbf{I}\ddot{\mathbf{z}} + \mathbf{A}\dot{\mathbf{z}} + \mathbf{B}\mathbf{z} \quad = \quad \mathbf{0}$$

Where:

$$\begin{array}{lcl}\mathbf{A} & \overset{\text{def}}{=} & [\boldsymbol{\Theta}]^T\mathbf{c}[\boldsymbol{\Theta}] \\ \mathbf{B} & \overset{\text{def}}{=} & [\boldsymbol{\Theta}]^T\mathbf{k}[\boldsymbol{\Theta}]\end{array}$$

Since $\mathbf{k}$ & $\mathbf{c}$ are real symmetric and (at least) non-negative definite matrices, then, matrices $\mathbf{A}$ & $\mathbf{B}$ are also real symmetric and (at least) non-negative definite.

For the system $\mathbf{I}\ddot{\mathbf{z}} + \mathbf{A}\dot{\mathbf{z}} + \mathbf{B}\mathbf{z} = \mathbf{0}$ to possess classical normal modes then there must exist an **orthogonal transformation** $[\boldsymbol{\Psi}]$ (*i.e.*, $[\boldsymbol{\Psi}]^T[\boldsymbol{\Psi}] = \mathbf{I}$) that **reduces simultaneously to diagonal form the real symmetric matrices** $\mathbf{A}$ & $\mathbf{B}$. But such a transformation exists **if and only if** matrices $\mathbf{A}$ & $\mathbf{B}$ **commute** (BELLMAN, 1970, page 56); *i.e.*

$$\mathbf{AB} \quad = \quad \mathbf{BA}$$

BELLMAN, R. (1970/1997). *Introduction to Matrix Analysis*, Second Edition, SIAM

It is straightforward to obtain:

$$[\boldsymbol{\Theta}]^T\mathbf{m}[\boldsymbol{\Theta}] = \mathbf{I} \quad \Rightarrow \quad [\boldsymbol{\Theta}]^{-1}\mathbf{m}^{-1}([\boldsymbol{\Theta}]^T)^{-1} = \mathbf{I}^{-1} \quad \Rightarrow \quad \mathbf{m}^{-1} = [\boldsymbol{\Theta}][\boldsymbol{\Theta}]^T$$

Therefore, summarizing we have the identities:

$$\boxed{\begin{array}{lcl}\mathbf{A} & \overset{\text{def}}{=} & [\boldsymbol{\Theta}]^T\mathbf{c}[\boldsymbol{\Theta}] \\ \mathbf{B} & \overset{\text{def}}{=} & [\boldsymbol{\Theta}]^T\mathbf{k}[\boldsymbol{\Theta}] \\ \mathbf{m}^{-1} & = & [\boldsymbol{\Theta}][\boldsymbol{\Theta}]^T\end{array}}$$

Therefore, **if the matrices $\mathbf{A}$ & $\mathbf{B}$ commute**, *i.e.* if $\mathbf{AB} = \mathbf{BA}$, then:

$$\begin{aligned}
\mathbf{AB} &= \mathbf{BA}\\
[\boldsymbol{\Theta}]^T\mathbf{c}[\boldsymbol{\Theta}][\boldsymbol{\Theta}]^T\mathbf{k}[\boldsymbol{\Theta}] &= [\boldsymbol{\Theta}]^T\mathbf{k}[\boldsymbol{\Theta}][\boldsymbol{\Theta}]^T\mathbf{c}[\boldsymbol{\Theta}]\\
[\boldsymbol{\Theta}]^T\mathbf{c}\mathbf{m}^{-1}\mathbf{k}[\boldsymbol{\Theta}] &= [\boldsymbol{\Theta}]^T\mathbf{k}\mathbf{m}^{-1}\mathbf{c}[\boldsymbol{\Theta}]\\
\mathbf{c}\mathbf{m}^{-1}\mathbf{k} &= \mathbf{k}\mathbf{m}^{-1}\mathbf{c}\\
(\mathbf{m}^{-1}\mathbf{c})(\mathbf{m}^{-1}\mathbf{k}) &= (\mathbf{m}^{-1}\mathbf{k})(\mathbf{m}^{-1}\mathbf{c})
\end{aligned}$$

Therefore, in conclusion, the necessary and sufficient condition for classical normal modes to exist in $\mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = \mathbf{0}$, is that $\mathbf{cm^{-1}k} = \mathbf{km^{-1}c}$ or, equivalently, that **matrices** $(\mathbf{m^{-1}c})$ & $(\mathbf{m^{-1}k})$ **commute**.

As was stated above, let the orthogonal transformation that reduces **simultaneously** to diagonal form the real symmetric matrices **A** & **B** be the following:

$$\mathbf{z}(t) = [\boldsymbol{\Psi}]\,\mathbf{q}(t)$$

Therefore:

$$\left.\begin{aligned}\mathbf{u}(t) &= [\boldsymbol{\Theta}]\,\mathbf{z}(t)\\ \mathbf{z}(t) &= [\boldsymbol{\Psi}]\,\mathbf{q}(t)\end{aligned}\right\} \Rightarrow \quad \mathbf{u}(t) = \underbrace{[\boldsymbol{\Theta}]\,[\boldsymbol{\Psi}]}_{[\boldsymbol{\Phi}]}\,\mathbf{q}(t) \;\Downarrow\; \boxed{\mathbf{u}(t) = [\boldsymbol{\Phi}]\,\mathbf{q}(t)}$$

Summarizing:

$$\begin{aligned}
[\boldsymbol{\Phi}]^T\mathbf{m}[\boldsymbol{\Phi}] &= [\boldsymbol{\Psi}]^T[\boldsymbol{\Theta}]^T\mathbf{m}[\boldsymbol{\Theta}]\,[\boldsymbol{\Psi}] = [\boldsymbol{\Psi}]^T\mathbf{I}\,[\boldsymbol{\Psi}] = [\boldsymbol{\Psi}]^T\,[\boldsymbol{\Psi}] = \mathbf{I}\\
[\boldsymbol{\Phi}]^T\mathbf{c}[\boldsymbol{\Phi}] &= [\boldsymbol{\Psi}]^T[\boldsymbol{\Theta}]^T\mathbf{c}[\boldsymbol{\Theta}]\,[\boldsymbol{\Psi}] = [\boldsymbol{\Psi}]^T\mathbf{A}\,[\boldsymbol{\Psi}]\\
[\boldsymbol{\Phi}]^T\mathbf{k}[\boldsymbol{\Phi}] &= [\boldsymbol{\Psi}]^T[\boldsymbol{\Theta}]^T\mathbf{k}[\boldsymbol{\Theta}]\,[\boldsymbol{\Psi}] = [\boldsymbol{\Psi}]^T\mathbf{B}\,[\boldsymbol{\Psi}]
\end{aligned}$$

If classical normal modes exist (and the necessary and sufficient condition for this is: $\mathbf{cm^{-1}k} = \mathbf{km^{-1}c}$ or, equivalently, that **matrices** $(\mathbf{m^{-1}c})$ & $(\mathbf{m^{-1}k})$ **commute**), then:

$$[\boldsymbol{\Psi}]^T\mathbf{A}\,[\boldsymbol{\Psi}] = diag\begin{bmatrix}\ddots & & \\ & \lambda_A & \\ & & \ddots\end{bmatrix}$$

$$[\boldsymbol{\Psi}]^T\mathbf{B}\,[\boldsymbol{\Psi}] = diag\begin{bmatrix}\ddots & & \\ & \lambda_B & \\ & & \ddots\end{bmatrix}$$

Q.E.D.■

NOTE: Transformations of the form $(\mathbf{P\,A\,Q})$ are classified according to the restrictions imposed on the <u>non-singular</u> matrices **P** & **Q**. If $\mathbf{B} = (\mathbf{P\,A\,Q})$, then matrices **A** & **B** are called ***equivalent*** **matrices**.

Thus:

- If $\mathbf{P} = \mathbf{Q}^T$, the resulting transformation, $(\mathbf{Q}^T\ \mathbf{A\,Q})$, is called a ***congruence*** **transformation**.
- If $\mathbf{P} = \mathbf{Q}^{-1}$, the resulting transformation, $(\mathbf{Q}^{-1}\ \mathbf{A\,Q})$, is called a ***similarity*** **transformation**.
- If $\mathbf{P} = \mathbf{Q}^T = \mathbf{Q}^{-1}$, the resulting transformation, $(\mathbf{Q}^{-1}\ \mathbf{A\,Q}) = (\mathbf{Q}^T\ \mathbf{A\,Q})$, is called an ***orthogonal*** **transformation**.

This **terminology is motivated by certain <u>geometrical considerations</u>**. We notice that **<u>an '*orthogonal*' transformation is both a '*congruence*' and a '*similarity*' transformation</u>**.

FREE VIBRATION OF SYSTEMS WITH CLASSICAL DAMPING

$$\begin{array}{ll} \textit{Equation of Motion:} & \mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \\ \textit{Initial Conditions:} & \mathbf{u}_0 = \mathbf{u}(0) \quad , \quad \dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0) \end{array}$$

$$\textit{Classical Damping} \quad \Leftrightarrow \quad \mathbf{\Phi}^T \mathbf{c} \mathbf{\Phi} = \underbrace{\begin{bmatrix} \ddots & & \\ & 2\xi_n \omega_n M_n & \\ & & \ddots \end{bmatrix}}_{\textit{diagonal natrix}}$$

In the case of **MDOF system with *Classical Damping***, the **response/solution** can be conveniently expressed as a **superposition of the N mode shapes of the undamped system**.

Specifically, we formulate and solve the eigenvalue problem:

$$(\mathbf{k} - \lambda \mathbf{m})\boldsymbol{\phi} = \mathbf{0} \quad \Longrightarrow \quad \begin{cases} \lambda \overset{\text{def}}{=} \omega^2 \\ \boldsymbol{\phi}_i 's \end{cases}$$

Then, the response of the damped system is expressed as:

$$\mathbf{u}(t) = \sum_{n=1}^{N} q_n(t) \boldsymbol{\phi}_n$$

Substituting the above solution in the Equation of Motion **and using the <u>orthogonality</u> of the mode shapes** (*w.r.t.* **m**, **k**, & **c**), we obtain N ***uncoupled ODE's*** **in terms of the modal coordinates** $q_n(t)$, *i.e.*,

$$\ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = 0 \quad (n = 1,2,\cdots,N)$$

Solving the above N **uncoupled equations**, we obtain:

$$q_n(t) = e^{-\xi_n \omega_n t} \left[q_n(0) \cos(\omega_{Dn} t) + \frac{\dot{q}_n(0) + \xi_n \omega_n q_n(0)}{\omega_{Dn}} \sin(\omega_{Dn} t) \right]$$

$$\text{where:} \quad \omega_{Dn} = \omega_n \sqrt{1 - \xi_n^2} \quad , \quad q_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \mathbf{u}_0}{M_n} \quad , \quad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \dot{\mathbf{u}}_0}{M_n}$$

DYNAMIC ANALYSIS & RESPONSE OF LINEAR SYSTEMS

Undamped Linear Systems: Modal Analysis

$$\begin{array}{ll} \textit{Equation of Motion:} & \mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \\ \textit{Initial Conditions:} & \mathbf{u}_0 = \mathbf{u}(0) \quad , \quad \dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0) \end{array}$$

The solution of a linear undamped MDOF system can be expanded in terms of modal contributions (recall the *modal expansion theorem*):

$$\mathbf{u}(t) = \sum_{r=1}^{N} q_r(t)\boldsymbol{\phi}_r = \boldsymbol{\Phi}\mathbf{q}(t)$$

Substitution of the above expression of $\mathbf{u}(t)$ in the Equation of Motion gives:

$$\sum_{r=1}^{N}\{\mathbf{m}\boldsymbol{\phi}_r\ddot{q}_r(t)\} + \sum_{r=1}^{N}\{\mathbf{k}\boldsymbol{\phi}_r q_r(t)\} = \mathbf{p}(t)$$

Pre-multiplying by $\boldsymbol{\phi}_n^T$:

$$\sum_{r=1}^{N}\{(\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_r)\ddot{q}_r(t)\} + \sum_{r=1}^{N}\{(\boldsymbol{\phi}_n^T\mathbf{k}\boldsymbol{\phi}_r)q_r(t)\} = \boldsymbol{\phi}_n^T\mathbf{p}(t)$$

Invoking the **orthogonality** of $\boldsymbol{\phi}_n$'s (*w.r.t.* $\mathbf{m}$ & $\mathbf{k}$):

$$\underbrace{(\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n)}_{M_n}\ddot{q}_n(t) + \underbrace{(\boldsymbol{\phi}_n^T\mathbf{k}\boldsymbol{\phi}_n)}_{K_n}q_n(t) = \underbrace{\boldsymbol{\phi}_n^T\mathbf{p}(t)}_{P_n(t)}$$

$$\Longrightarrow \quad M_n\ddot{q}_n(t) + K_n q_n(t) = P_n(t) \quad (n = 1,2,\cdots,N)$$

$$\Longrightarrow \quad \ddot{q}_n(t) + \omega_n^2 q_n(t) = \frac{P_n(t)}{M_n}$$

Thus, we have to solve N **uncoupled equations** of the modal coordinates subject to the **initial conditions**:

$$q_n(0) = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0}{M_n} \quad , \quad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\dot{\mathbf{u}}_0}{M_n}$$

Damped Linear System with Classical Damping: Modal Analysis

$$\begin{array}{ll} \textit{Equation of Motion:} & \mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \\ \textit{Initial Conditions:} & \mathbf{u}_0 = \mathbf{u}(0) \quad , \quad \dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0) \end{array}$$

The solution of a linear undamped MDOF system can be expanded in terms of modal contributions (recall the *modal expansion theorem*):

$$\mathbf{u}(t) = \sum_{r=1}^{N} q_r(t)\boldsymbol{\phi}_r = \boldsymbol{\Phi}\mathbf{q}(t)$$

Substitution of the above expression of $\mathbf{u}(t)$ in the Equation of Motion gives:

$$\sum_{r=1}^{N}\{\mathbf{m}\boldsymbol{\phi}_r\ddot{q}_r(t)\} + \sum_{r=1}^{N}\{\mathbf{c}\boldsymbol{\phi}_r\dot{q}_r(t)\} + \sum_{r=1}^{N}\{\mathbf{k}\boldsymbol{\phi}_r q_r(t)\} = \mathbf{p}(t)$$

Pre-multiplying by $\boldsymbol{\phi}_n^T$:

$$\sum_{r=1}^{N}\{(\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_r)\ddot{q}_r(t)\} + \sum_{r=1}^{N}\{(\boldsymbol{\phi}_n^T\mathbf{c}\boldsymbol{\phi}_r)\dot{q}_r(t)\} + \sum_{r=1}^{N}\{(\boldsymbol{\phi}_n^T\mathbf{k}\boldsymbol{\phi}_r)q_r(t)\} = \boldsymbol{\phi}_n^T\mathbf{p}(t)$$

Invoking the **orthogonality** of $\boldsymbol{\phi}_n$'s (*w.r.t.* $\mathbf{m}$ & $\mathbf{k}$):

$$\underbrace{(\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n)}_{M_n}\ddot{q}_n(t) + \underbrace{(\boldsymbol{\phi}_n^T\mathbf{c}\boldsymbol{\phi}_n)}_{C_n}\dot{q}_r(t) + \underbrace{(\boldsymbol{\phi}_n^T\mathbf{k}\boldsymbol{\phi}_n)}_{K_n}q_n(t) = \underbrace{\boldsymbol{\phi}_n^T\mathbf{p}(t)}_{P_n(t)}$$

$$\Longrightarrow \quad M_n\ddot{q}_n(t) + C_n\dot{q}_n(t) + K_n q_n(t) = P_n(t) \qquad (n = 1,2,\cdots,N)$$

$$\Longrightarrow \quad \ddot{q}_n(t) + 2\xi_n\omega_n\dot{q}_n(t) + \omega_n^2 q_n(t) = \frac{P_n(t)}{M_n}$$

Thus, we have to solve N **uncoupled equations** of the modal coordinates subject to the **initial conditions**:

$$q_n(0) = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0}{M_n} \quad , \quad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\dot{\mathbf{u}}_0}{M_n}$$

ADDITIONAL ORTHOGONALITY CONDITIONS

Before establishing the conditions for damping orthogonality, we need to develop **a series of additional mode shape orthogonality conditions** similar to those given by:

$$\boldsymbol{\phi}_i^T \mathbf{m} \boldsymbol{\phi}_j = 0 \quad , \quad \boldsymbol{\phi}_i^T \mathbf{k} \boldsymbol{\phi}_j = 0 \quad \lambda_i \neq \lambda_j$$

We start with the eigenvalue problem:

$$\mathbf{k}\boldsymbol{\phi}_i = \lambda_i \mathbf{m}\boldsymbol{\phi}_i$$

$$\overbrace{\boldsymbol{\phi}_j^T \mathbf{k}\mathbf{m}^{-1}}\mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_j^T \mathbf{k}\mathbf{m}^{-1}}\mathbf{m}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_j^T \mathbf{k}\boldsymbol{\phi}_i = 0 \quad , \quad (i \neq j)$$

$$\overbrace{\boldsymbol{\phi}_i^T \mathbf{k}\mathbf{m}^{-1}}\mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_i^T \mathbf{k}\mathbf{m}^{-1}}\mathbf{m}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i^T \mathbf{k}\boldsymbol{\phi}_i = \lambda_i^2 \quad , \quad (i = j)$$

[assuming $\boldsymbol{\phi}_i$'s are ***orthonormal*** $\Leftrightarrow \boldsymbol{\phi}_i^T \mathbf{m}\boldsymbol{\phi}_i = 1$]

Similarly:

$$\overbrace{\boldsymbol{\phi}_j^T \mathbf{k}\mathbf{m}^{-1}\mathbf{k}\mathbf{m}^{-1}}\mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_j^T \mathbf{k}\mathbf{m}^{-1}\mathbf{k}\mathbf{m}^{-1}}\mathbf{m}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_j^T \mathbf{k}\mathbf{m}^{-1}\mathbf{k}\boldsymbol{\phi}_i = 0 \quad , \quad (i \neq j)$$

$$\overbrace{\boldsymbol{\phi}_i^T \mathbf{k}\mathbf{m}^{-1}\mathbf{k}\mathbf{m}^{-1}}\mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_i^T \mathbf{k}\mathbf{m}^{-1}\mathbf{k}\mathbf{m}^{-1}}\mathbf{m}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i^T \mathbf{k}\mathbf{m}^{-1}\mathbf{k}\boldsymbol{\phi}_i = \lambda_i^3 \quad , \quad (i = j)$$

In general:

$$\boxed{\boldsymbol{\phi}_j^T (\mathbf{k}\mathbf{m}^{-1})^a \mathbf{k}\boldsymbol{\phi}_i = 0 \quad \begin{Bmatrix} i \neq j \\ a = 0,1,2,\cdots,\infty \end{Bmatrix} \quad , \quad \boldsymbol{\phi}_i^T (\mathbf{k}\mathbf{m}^{-1})^a \mathbf{k}\boldsymbol{\phi}_i = \lambda_i^{a+1} \quad \begin{Bmatrix} i = j \\ a = 0,1,2,\cdots,\infty \end{Bmatrix}}$$

Then, it follows that:

$$\boldsymbol{\phi}_j^T (\mathbf{k}\mathbf{m}^{-1})^a \mathbf{k}\boldsymbol{\phi}_i = 0 \quad \begin{Bmatrix} i \neq j \\ a = 0,1,2,\cdots,\infty \end{Bmatrix} \rightarrow \begin{cases} \Rightarrow & \boldsymbol{\phi}_j^T \overbrace{\mathbf{m}\mathbf{m}^{-1}}^{\mathbf{I}} (\mathbf{k}\mathbf{m}^{-1})^a \mathbf{k}\boldsymbol{\phi}_i = 0 \\ \Rightarrow & \boldsymbol{\phi}_j^T \mathbf{m}\mathbf{m}^{-1} \underbrace{(\mathbf{k}\mathbf{m}^{-1})(\mathbf{k}\mathbf{m}^{-1})\cdots(\mathbf{k}\mathbf{m}^{-1})}_{a\ times} \mathbf{k}\boldsymbol{\phi}_i = 0 \\ \Rightarrow & \boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{a+1}\boldsymbol{\phi}_i = 0 \end{cases}$$

Similarly:

$$\boldsymbol{\phi}_i^T (\mathbf{k}\mathbf{m}^{-1})^a \mathbf{k}\boldsymbol{\phi}_i = \lambda_i^{a+1} \quad \begin{Bmatrix} i = j \\ a = 0,1,2,\cdots,\infty \end{Bmatrix} \rightarrow \begin{cases} \Rightarrow & \boldsymbol{\phi}_i^T \overbrace{\mathbf{m}\mathbf{m}^{-1}}^{\mathbf{I}} (\mathbf{k}\mathbf{m}^{-1})^a \mathbf{k}\boldsymbol{\phi}_i = \lambda_i^{a+1} \\ \Rightarrow & \boldsymbol{\phi}_i^T \mathbf{m}\mathbf{m}^{-1} \underbrace{(\mathbf{k}\mathbf{m}^{-1})(\mathbf{k}\mathbf{m}^{-1})\cdots(\mathbf{k}\mathbf{m}^{-1})}_{a\ times} \mathbf{k}\boldsymbol{\phi}_i = \lambda_i^{a+1} \\ \Rightarrow & \boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{a+1}\boldsymbol{\phi}_i = \lambda_i^{a+1} \end{cases}$$

In general:

$$\boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^b \boldsymbol{\phi}_i = 0 \quad \begin{cases} i \neq j \\ b = 0,1,2,\cdots,\infty \end{cases}$$

$$\boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^b \boldsymbol{\phi}_i = \lambda_i^b \quad \begin{cases} i = j \\ b = 0,1,2,\cdots,\infty \end{cases}$$

[assuming the $\boldsymbol{\phi}_i$'s are ***orthonormal*** w.r.t. $\mathbf{m} \Leftrightarrow \boldsymbol{\phi}_i^T \mathbf{m}\boldsymbol{\phi}_i = 1$]

We start again with the eigenvalue problem:

$$\mathbf{k}\boldsymbol{\phi}_i = \lambda_i \mathbf{m}\boldsymbol{\phi}_i \qquad (1)$$

$$(1) \Rightarrow \overbrace{\boldsymbol{\phi}_j^T \mathbf{m}\mathbf{k}^{-1}}\, \mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_j^T \mathbf{m}\mathbf{k}^{-1}}\, \mathbf{m}\boldsymbol{\phi}_i \Rightarrow \underbrace{\boldsymbol{\phi}_j^T \mathbf{m}\boldsymbol{\phi}_i}_{0} = \lambda_i \boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-1}\boldsymbol{\phi}_i$$

$$\therefore \quad \boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-1}\boldsymbol{\phi}_i = 0 \quad \begin{cases} i \neq j \\ \lambda_i \neq 0 \end{cases}$$

$$(1) \Rightarrow \overbrace{\boldsymbol{\phi}_i^T \mathbf{m}\mathbf{k}^{-1}}\, \mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_i^T \mathbf{m}\mathbf{k}^{-1}}\, \mathbf{m}\boldsymbol{\phi}_i \Rightarrow \underbrace{\boldsymbol{\phi}_i^T \mathbf{m}\boldsymbol{\phi}_i}_{1} = \lambda_i \boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-1}\boldsymbol{\phi}_i$$

$$[mode\ shapes\ are\ orthonormal\ w.r.t.\,\mathbf{m}]$$

$$\therefore \quad \boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-1}\boldsymbol{\phi}_i = \lambda_i^{-1} \quad \begin{cases} i = j \\ \lambda_i \neq 0 \end{cases}$$

Similarly:

$$(1) \Rightarrow \overbrace{\boldsymbol{\phi}_j^T \mathbf{m}\mathbf{k}^{-1}\mathbf{m}\mathbf{k}^{-1}}\, \mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_j^T \mathbf{m}\mathbf{k}^{-1}\mathbf{m}\mathbf{k}^{-1}}\, \mathbf{m}\boldsymbol{\phi}_i \Rightarrow \underbrace{\boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-1}\boldsymbol{\phi}_i}_{0} = \lambda_i \boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-2}\boldsymbol{\phi}_i$$

$$\therefore \quad \boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-2}\boldsymbol{\phi}_i = 0 \quad \begin{cases} i \neq j \\ \lambda_i \neq 0 \end{cases}$$

$$(1) \Rightarrow \overbrace{\boldsymbol{\phi}_i^T \mathbf{m}\mathbf{k}^{-1}\mathbf{m}\mathbf{k}^{-1}}\, \mathbf{k}\boldsymbol{\phi}_i = \lambda_i \overbrace{\boldsymbol{\phi}_i^T \mathbf{m}\mathbf{k}^{-1}\mathbf{m}\mathbf{k}^{-1}}\, \mathbf{m}\boldsymbol{\phi}_i \Rightarrow \underbrace{\boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-1}\boldsymbol{\phi}_i}_{\lambda_i^{-1}} = \lambda_i \boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-2}\boldsymbol{\phi}_i$$

$$\therefore \quad \boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{-2}\boldsymbol{\phi}_i = \lambda_i^{-2} \quad \begin{cases} i = j \\ \lambda_i \neq 0 \end{cases}$$

Thus, by induction we obtain:

$$\boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{b}\boldsymbol{\phi}_i = 0 \quad \begin{Bmatrix} i \neq j \quad \lambda_i \neq 0 \\ b = -1, -2, \cdots, -\infty \end{Bmatrix} \quad , \quad \boldsymbol{\phi}_i^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{b}\boldsymbol{\phi}_i = \lambda_i^{b} \quad \begin{Bmatrix} i = j \quad \lambda_i \neq 0 \\ b = -1, -2, \cdots, -\infty \end{Bmatrix}$$

Thus, combining the above expressions for b **negative** with the expressions we derived above for b **positive**, we get:

$$\boldsymbol{\phi}_j^T \mathbf{m}(\mathbf{m}^{-1}\mathbf{k})^{b}\boldsymbol{\phi}_i = \begin{cases} 0 & i \neq j \\ \lambda_i^{b} & i = j \end{cases} \quad -\infty < b < +\infty$$

$$[mode\ shapes\ are\ orthonormal\ w.r.t.\,\mathbf{m} \Leftrightarrow \boldsymbol{\phi}_i^T \mathbf{m}\boldsymbol{\phi}_i = 1]$$

CLASSICAL DAMPING MATRIX

Classical damping matrix is an appropriate idealization **if similar damping mechanisms throughout the structure** (*e.g.*, multi-story building with a similar structural system and structural materials over its height).

It should be noted that when a system is assumed to possess ***proportional damping***, the mode superposition method can be used in the analysis and the damping matrix is not required, provided that a **damping ratio** can be specified **for each mode** that has been included.

It would therefore appear that there is no need to specify a damping matrix explicitly for a system possessing classical damping.

There, are however, **situations when a damping matrix is required**.

For example, **for the calculation of the response of structures beyond their linearly elastic range during earthquakes**, numerical time integration must be used and the damping matrix has to be specified.

When a **(classical) damping matrix** is required, **it should be constructed in such a manner that it would lead to specified values for the damping ratio in some or all of the modes**. The construction of such a matrix requires a study of **the conditions of damping orthogonality**.

TABLE RECOMMENDED DAMPING VALUES

Stress Level	Type and Condition of Structure	Damping Ratio (%)
Working stress, no more than about $\frac{1}{2}$ yield point	Welded steel, prestressed concrete, well-reinforced concrete (only slight cracking)	2–3
	Reinforced concrete with considerable cracking	3–5
	Bolted and/or riveted steel, wood structures with nailed or bolted joints	5–7
At or just below yield point	Welded steel, prestressed concrete (without complete loss in prestress)	5–7
	Prestressed concrete with no prestress left	7–10
	Reinforced concrete	7–10
	Bolted and/or riveted steel, wood structures with bolted joints	10–15
	Wood structures with nailed joints	15–20

Source: N. M. Newmark, and W. J. Hall, *Earthquake Spectra and Design*, Earthquake Engineering Research Institute, Berkeley, Calif., 1982.

RAYLEIGH DAMPING

We can use the orthogonality relationships derived previously to develop a damping matrix that will satisfy the condition of orthogonality.

Mass Proportional Damping:

$$\boxed{\mathbf{c} = \alpha_0 \mathbf{m} \quad (units\ of\ [\alpha_0] = [T]^{-1})}$$

Therefore:

$$\left.\begin{aligned} \boldsymbol{\phi}_j^T \mathbf{c} \boldsymbol{\phi}_i &= \alpha_0 \underbrace{\boldsymbol{\phi}_j^T \mathbf{m} \boldsymbol{\phi}_i}_{0} \\ &= 0 \end{aligned}\right\} (i \neq j) \quad , \quad \left.\begin{aligned} &\underbrace{\boldsymbol{\phi}_i^T \mathbf{c} \boldsymbol{\phi}_i}_{C_i} = \alpha_0 \underbrace{\boldsymbol{\phi}_i^T \mathbf{m} \boldsymbol{\phi}_i}_{M_i} \\ &\left.\begin{aligned} C_i &= \alpha_0 M_i \\ C_i &\stackrel{\text{def}}{=} 2\xi_i \omega_i M_i \end{aligned}\right\} \Rightarrow \boxed{\alpha_0 = 2\xi_i \omega_i} \end{aligned}\right\} (i = j)$$

Once a value of α_0 has been selected as above, the damping in any other mode will be given by:

$$\boxed{\xi_j = \frac{\alpha_0}{2\omega_j} = \xi_i \frac{\omega_i}{\omega_j}}$$

Stiffness Proportional Damping:

$$\boxed{\mathbf{c} = \alpha_1 \mathbf{k} \quad (units\ of\ [\alpha_1] = [T])}$$

Therefore:

$$\left.\begin{aligned} \boldsymbol{\phi}_j^T \mathbf{c} \boldsymbol{\phi}_i &= \alpha_1 \underbrace{\boldsymbol{\phi}_j^T \mathbf{k} \boldsymbol{\phi}_i}_{0} \\ &= 0 \end{aligned}\right\} (i \neq j) \quad , \quad \left.\begin{aligned} &\underbrace{\boldsymbol{\phi}_i^T \mathbf{c} \boldsymbol{\phi}_i}_{C_i} = \alpha_1 \underbrace{\boldsymbol{\phi}_i^T \mathbf{k} \boldsymbol{\phi}_i}_{\alpha_1 \omega_i^2 M_i} \\ &\left.\begin{aligned} C_i &= \alpha_1 \omega_i^2 M_i \\ C_i &\stackrel{\text{def}}{=} 2\xi_i \omega_i M_i \end{aligned}\right\} \Rightarrow \boxed{\alpha_1 = \frac{2\xi_i}{\omega_i}} \end{aligned}\right\} (i = j)$$

Having selected a value of $\boldsymbol{\alpha_1}$ as above, **the damping ratio for any other mode can be determined**:

$$\boxed{\xi_j = \frac{1}{2}\omega_j \alpha_1 = \xi_i \frac{\omega_j}{\omega_i}}$$

Rayleigh Damping:

As a more general case, we can select the damping matrix to be a **linear combination** of the mass and the stiffness matrices, so that:

$$\boxed{\mathbf{c} = \alpha_0 \mathbf{m} + \alpha_1 \mathbf{k}}$$

The above equation has **two free parameters**, α_0 & α_1.

We can thus **specify the damping ratio for any two modes**, say the i^{th} and j^{th}.

Then:

$$\boldsymbol{\phi}_i^T \mathbf{c} \boldsymbol{\phi}_i = 2\xi_i \omega_i M_i = \alpha_0 M_i + \alpha_1 \omega_i^2 M_i$$
$$\boldsymbol{\phi}_j^T \mathbf{c} \boldsymbol{\phi}_j = 2\xi_j \omega_j M_j = \alpha_0 M_j + \alpha_1 \omega_j^2 M_j$$

or, in matrix form:

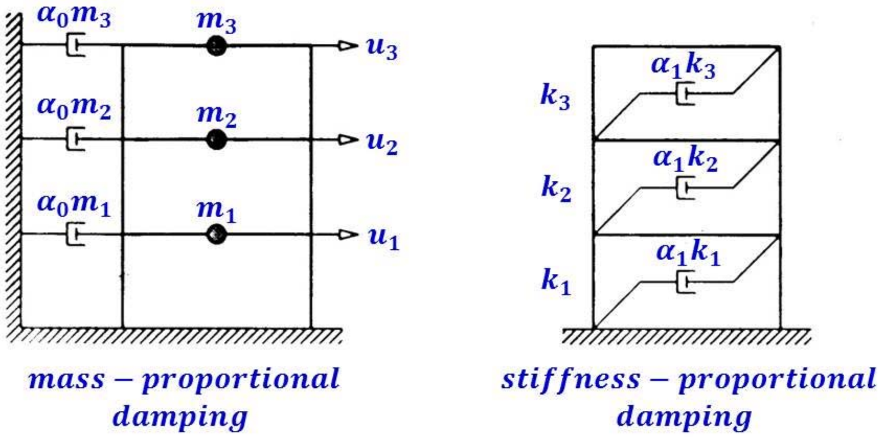
$$\boxed{\frac{1}{2}\begin{pmatrix} \dfrac{1}{\omega_i} & \omega_i \\ \dfrac{1}{\omega_j} & \omega_j \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \xi_i \\ \xi_j \end{pmatrix}}$$

The above equation can be solved for α_0 & α_1 :

$$\alpha_0 = \frac{2\omega_i \omega_j \left(\xi_i \omega_j - \xi_j \omega_i\right)}{\left(\omega_j^2 - \omega_i^2\right)} \quad , \quad \alpha_1 = \frac{2\left(\xi_j \omega_j - \xi_i \omega_i\right)}{\left(\omega_j^2 - \omega_i^2\right)}$$

The damping ratio of any other mode, say the n^{th} mode, is:

$$\boxed{\xi_n = \frac{1}{2}\left(\alpha_0 \frac{1}{\omega_n} + \alpha_1 \omega_n\right)}$$



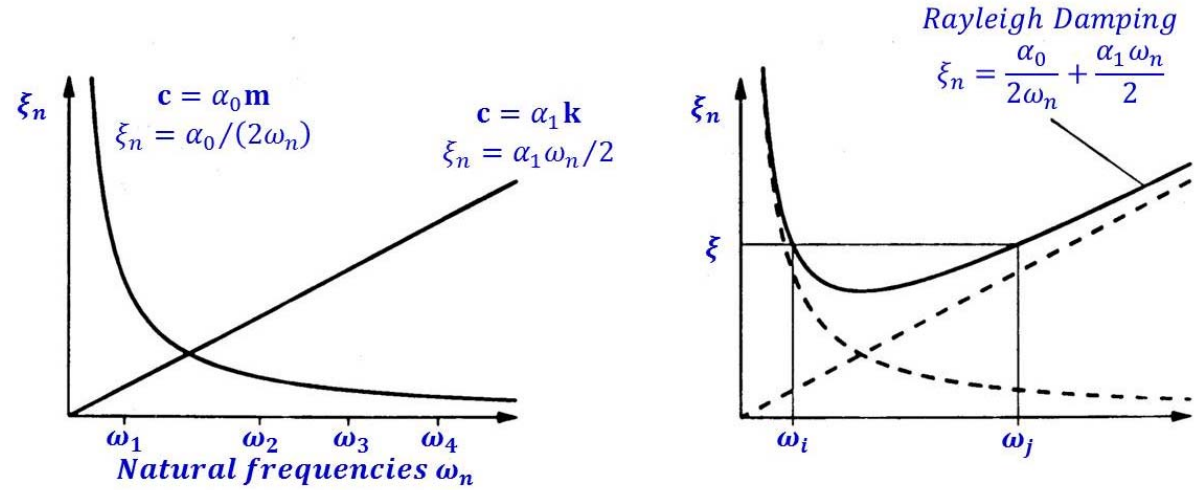


FIGURE: Variation of modal damping ratios with natural frequency: (a) mass-proportional damping and stiffness-proportional damping; (b) Rayleigh damping.

Caughey (or Extended Rayleigh) Damping:

The mass and stiffness matrices, $\mathbf{m}$ & $\mathbf{k}$ respectively, used to formulate *Rayleigh* damping are not the only matrices to which the orthogonality relations of the (undamped system) mode-shapes apply; in fact, it has been demonstrated earlier that an infinite number of matrices have this property.

Therefore, a classical damping matrix can be made up of any combination of these matrices, as follows:

$$\mathbf{c} = \mathbf{m}\sum_{\ell} \alpha_{\ell}[\mathbf{m}^{-1}\mathbf{k}]^{\ell} = \sum_{\ell} \mathbf{c}_{\ell} \qquad \ell = \cdots, -2, -1, 0, +1, +2, \cdots \tag{1}$$

It is evident that **Rayleigh damping is given by Equation (1) above if only the terms $\ell = 0$ and $\ell = 1$ are retained in the series**.

We have demonstrated previously that:

$$\boldsymbol{\phi}_j^T \mathbf{m}[\mathbf{m}^{-1}\mathbf{k}]^{\ell}\boldsymbol{\phi}_j = \omega_j^{2\ell} M_j$$

Therefore:

$$\left.\begin{aligned} \underbrace{\boldsymbol{\phi}_j^T \mathbf{c} \boldsymbol{\phi}_j}_{C_j} &= \sum_{\ell} \alpha_{\ell}\omega_j^{2\ell} M_j \\ C_j = &= 2\xi_j\omega_j M_j \end{aligned}\right\} \Rightarrow \boxed{\xi_j = \frac{1}{2\omega_j}\sum_{\ell} \alpha_{\ell}\omega_j^{2\ell}}$$

The above equation **provides the means for evaluating the constants α_{ℓ} to give the desired damping at any specified number** (up to N for N-DOF system) of **modal frequencies**.

As many terms must be included in the series as there are specified modal damping ratios; then the constants are given by the solution of the set of equations, one written for each damping ratio.

In principle, the values of ℓ can be anywhere in the range $-\infty < \ell < +\infty$, but **in practice it is desirable to select values of these exponents as close to zero as possible**.

For example, to evaluate the coefficients that will provide specified damping ratios in any four modes having frequencies $\omega_m, \omega_n, \omega_o, \omega_p$ the equations resulting from:

$$\xi_j = \frac{1}{2\omega_j}\sum_{\ell} \alpha_\ell \omega_j^{2\ell}$$

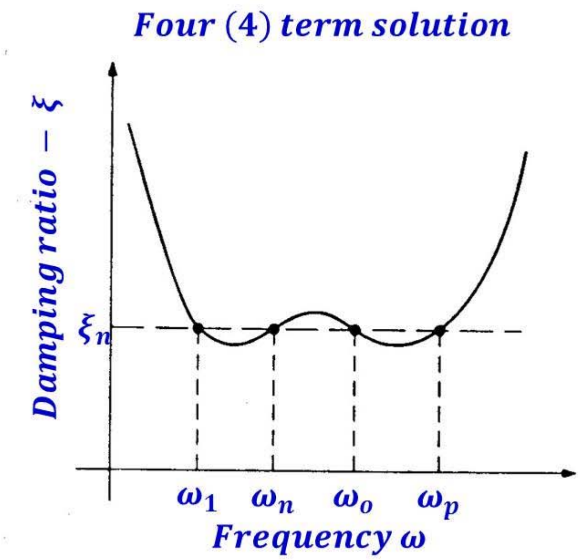
Using the terms for $\ell = -1, 0, +1, +2$ are:

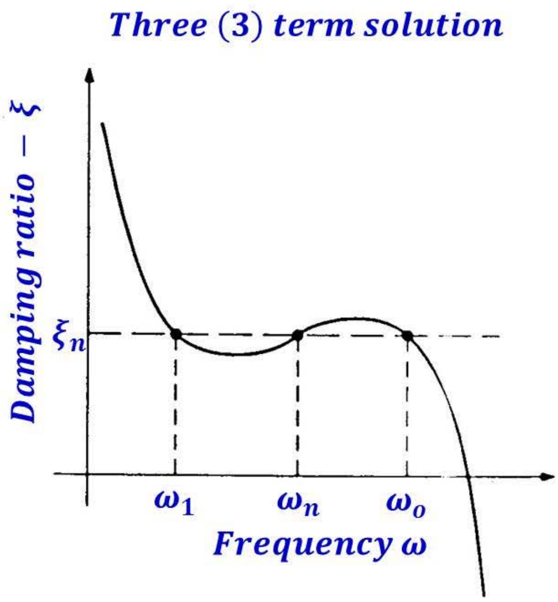
$$\begin{pmatrix} \xi_m \\ \xi_n \\ \xi_o \\ \xi_p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1/\omega_m^2 & 1/\omega_m & \omega_m & \omega_m^3 \\ 1/\omega_n^2 & 1/\omega_n & \omega_n & \omega_n^3 \\ 1/\omega_o^2 & 1/\omega_o & \omega_o & \omega_o^3 \\ 1/\omega_p^2 & 1/\omega_p & \omega_p & \omega_p^3 \end{pmatrix}\begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}$$

When the coefficients $\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2$ have been evaluated by solving the above system of equations, the classical damping matrix that provides the four required damping ratios at the four specified frequencies is:

$$\mathbf{c} = \mathbf{m}\{\alpha_{-1}[\mathbf{m}^{-1}\mathbf{k}]^{-1} + \alpha_0\mathbf{I} + \alpha_1[\mathbf{m}^{-1}\mathbf{k}] + \alpha_2[\mathbf{m}^{-1}\mathbf{k}]^2\}$$

FIGURE (a) below illustrates the relation between **damping ratio** and **frequency** that would result from this matrix.





To simplify FIGURE (a) it has been assumed that the same damping ratio, ξ_x , was specified for all four (4) frequencies; however, each of the damping ratios could have been specified arbitrarily.

Also, ω_m has been taken as the **fundamental mode frequency**, ω_1, and ω_p is intended to approximate the **frequency of the highest mode that contributes significantly to the response**, while ω_n and ω_o are **spaced about equally within the frequency range**.

It is evident in FIGURE (a) that the damping ratio remains close to the desired value ξ_x throughout the frequency range, being exact at the four specified frequencies and ranging slightly above or below at other frequencies in the range. It is important to note, however, that the damping increases monotonically with frequency for frequencies increasing above ω_p. This has the effect of excluding any significant contribution from any modes with frequencies much greater than ω_p.

An even more important point to note is the consequence of including only three (3) terms in the derivation of the damping matrix $\mathbf{c}$ [FIGURE (b)].

The serious defect of this case is that the damping decreases monotonically for $\omega_o < \omega$ and negative damping is indicated for all the highest modal frequencies. This is physically unacceptable.

The general implication of this observation is that CAUGHEY damping may be used effectively only if an <u>even</u> number of terms is included in the series expression of $\mathbf{c}$.