FREE VIBRATION RESPONSE: MDOF SYSTEMS

SYSTEMS WITHOUT DAMPING

Equation of Motion: $$\boxed{\mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{0}}$$

A very important case in the study of vibrations of MDOF systems is that in which all the coordinates, *i.e.*, **all components** $u_i(t)$ $(i = 1,2,\cdots,N)$ of $\mathbf{u}(t)$, **execute the same motion in time**.

In other words, **the system vibrates maintaining the overall shape of these coordinates/displacements/deflections and changing only their amplitude by a proportionality factor**.

In this case, the system is said to execute ***synchronous motion***.

To examine the possibility that such motions exist, we consider the solution of the Equation of Motion (see above) in the exponential form:

$$\boxed{\begin{array}{c} \mathbf{u}(t) = e^{st}\boldsymbol{\phi} \\ \text{where: } \begin{cases} s & \text{is a constant scalar} \\ \boldsymbol{\phi} & \text{is a constant } N\text{-vector} \end{cases} \end{array}}$$

Introducing the solution $\mathbf{u}(t) = e^{st}\boldsymbol{\phi}$ in the Equation of Motion we obtain:

$$\left.\begin{array}{r} \mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{0} \\ \mathbf{u}(t) = e^{st}\boldsymbol{\phi} \end{array}\right\} \quad \Longrightarrow \quad e^{st}(s^2\mathbf{m}\boldsymbol{\phi} + \mathbf{k}\boldsymbol{\phi}) = \mathbf{0} \quad \Longrightarrow \quad \mathbf{k}\boldsymbol{\phi} = -s^2\mathbf{m}\boldsymbol{\phi}$$

Therefore:

$$\boxed{\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi} \quad , \quad \lambda = -s^2}$$

The above matrix equation represents a set of N **simultaneous homogeneous algebraic equations** in the unknowns $\boldsymbol{\phi} \overset{\text{def}}{=} \{\phi_i\}$ $(i = 1,2,\cdots,N)$, with λ **playing the role of a parameter**.

The problem of **determining the values of the parameter** λ for which the matrix equation $\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi}$ admits **nontrivial** solutions $\boldsymbol{\phi}$ is known as the ***algebraic*** (or ***matrix***) ***eigenvalue*** (or ***characteristic-value***) ***problem***. [**NOTE**: The obvious, but trivial, solution is $\boldsymbol{\phi} = \mathbf{0}$, however we are not interested in such a solution because it is not associated with motion.] Another name by which the above problem may be encountered is ***generalized characteristic-value problem***.

Before we proceed, we summarize the properties of the matrix eigenvalue problem that are demonstrated mathematically:

PROPERTIES OF THE ALGEBRAIC (or MATRIX) EIGENVALUE PROBLEM

(1) **The eigenvalues** of an algebraic eigenvalue problem, in which $\mathbf{k}$ & $\mathbf{m}$ are **both symmetric**, and at **least one positive definite**, are **all real**.

[**NOTE**: For civil engineering structures, both matrices $\mathbf{k}$ & $\mathbf{m}$ are **real**, **symmetric** and **positive definite**. Indeed, $\mathbf{u}^T\mathbf{k}\mathbf{u} > 0$ for $\mathbf{u} \neq \mathbf{0}$, because civil engineering structures are **restrained** and **stable**, and $\mathbf{u}^T\mathbf{m}\mathbf{u} > 0$ for $\mathbf{u} \neq \mathbf{0}$, because **all DOFs associated with zero mass/inertia have been eliminated by static condensation**.]

(2) **When matrices $\mathbf{k}$ & $\mathbf{m}$** are **both real** and **positive definite**, **the eigenvalues are all positive**, and the **corresponding eigenvectors are all real**.

(3) When $\mathbf{k}$ is **singular, at least one** of the eigenvalues must be **zero**.
When $\mathbf{m}$ is **singular, at least one** of the eigenvalues must be **infinite**.

(4) The **eigenvectors are orthogonal with respect** to both $\mathbf{k}$ & $\mathbf{m}$.
For **symmetric** $\mathbf{k}$ & $\mathbf{m}$ matrices:

$$\boldsymbol{\phi}_i^T\mathbf{m}\boldsymbol{\phi}_j = 0 \quad i \neq j$$
$$\boldsymbol{\phi}_i^T\mathbf{k}\boldsymbol{\phi}_j = 0 \quad i \neq j$$

(5) **The eigenvectors** of an algebraic eigenvalue problem in which the **matrices involved are symmetric**, including those corresponding to repeated eigenvalues, **are all *linearly independent***.

(6) **Any arbitrary vector of order** N can be expressed as a **superposition** of the eigenvectors of an $N \times N$ symmetrical algebraic eigenvalue problem:

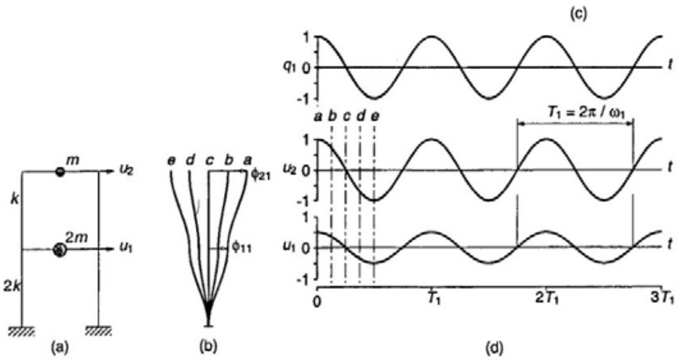
$$\mathbf{u} = \sum_{r=1}^{N} \boldsymbol{\phi}_r q_r = \boldsymbol{\Phi}\mathbf{q}$$

$$\text{where:} \quad \boldsymbol{\Phi} = \underbrace{\begin{pmatrix} \downarrow & \downarrow & & \downarrow \\ \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{\textit{Modal Matrix}}$$

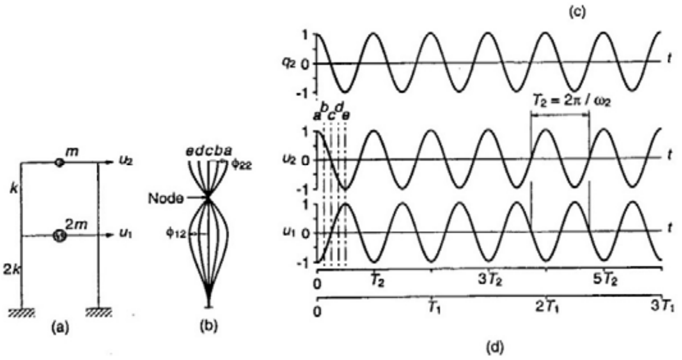
where $q_r \ (r = 1,2,\cdots,N)$ are scalar multipliers called ***modal coordinates*** or ***normal coordinates*** and $\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_N]^T$. This theorem is referred to as ***modal expansion theorem*** or ***eigenvector expansion theorem***.

Examples of **models of structures** and **their modal shapes**:

Free vibration of undamped system: $\mathbf{u}(0) = \phi_1$



Free vibration of undamped system: $\mathbf{u}(0) = \phi_2$



Free vibration of undamped system: Arbitrary $\mathbf{u}(0)$

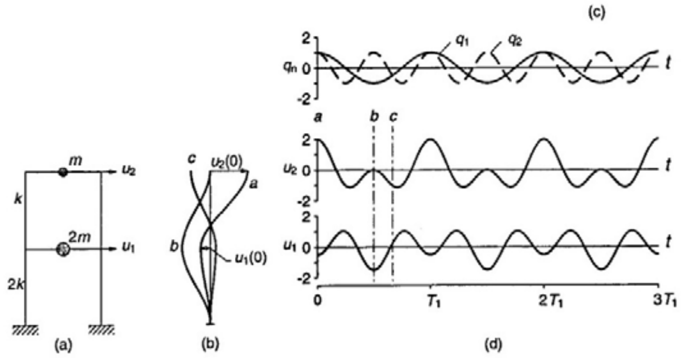
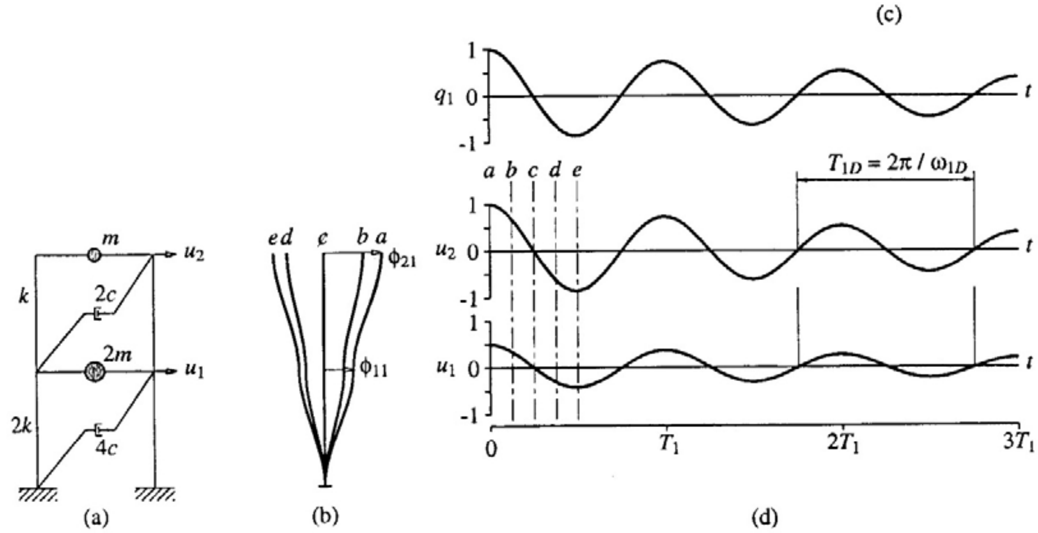


FIGURE: Free vibration of an undamped 2-DOF system.



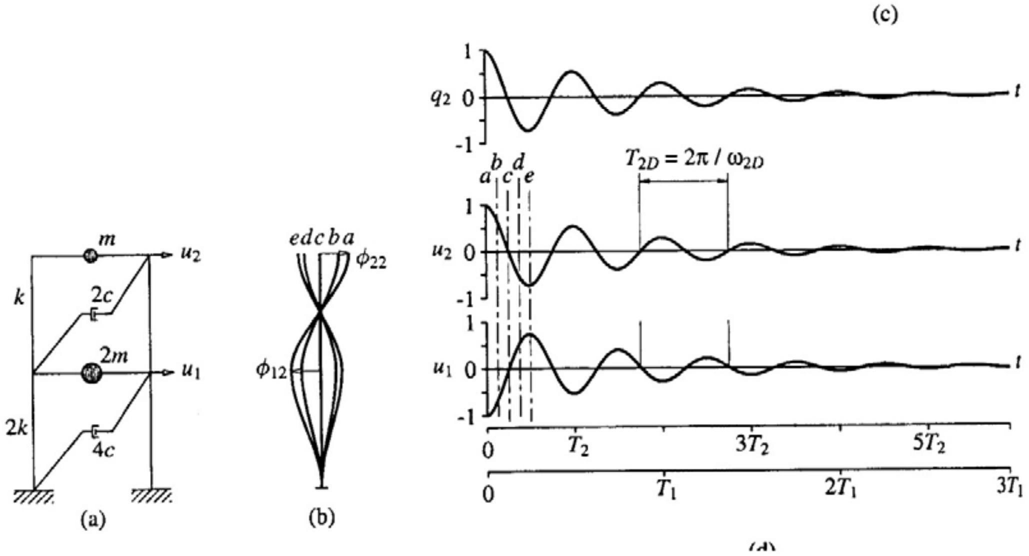


FIGURE: Free vibration of a classically damped 2-DOF system.

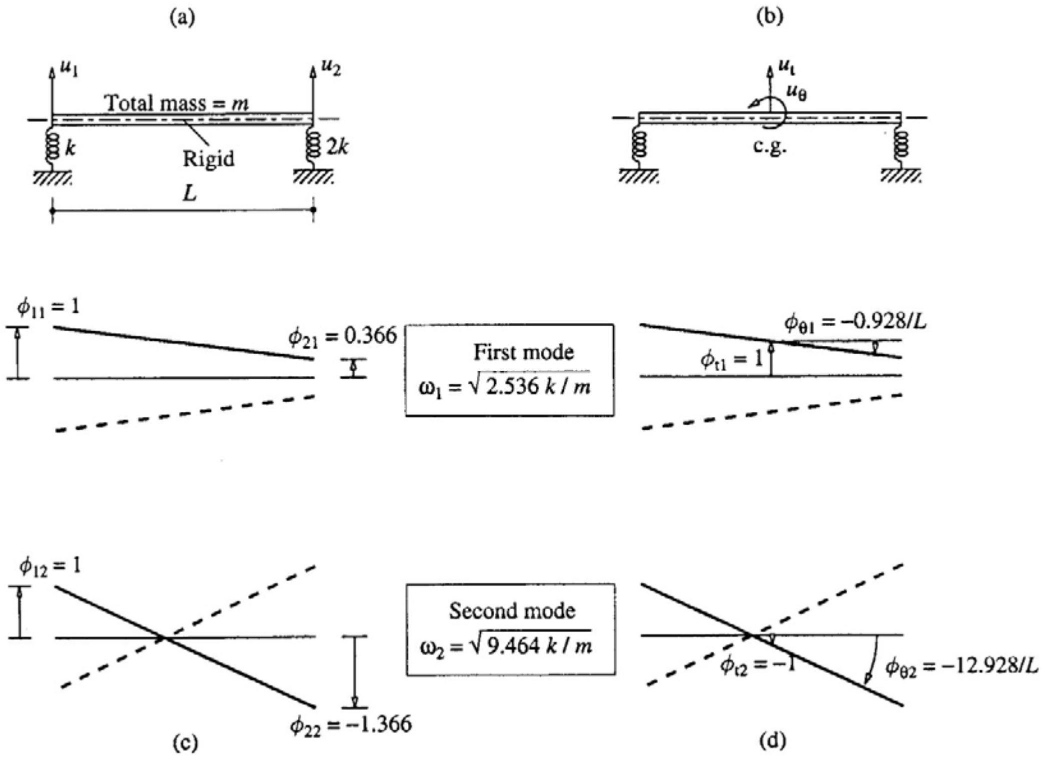
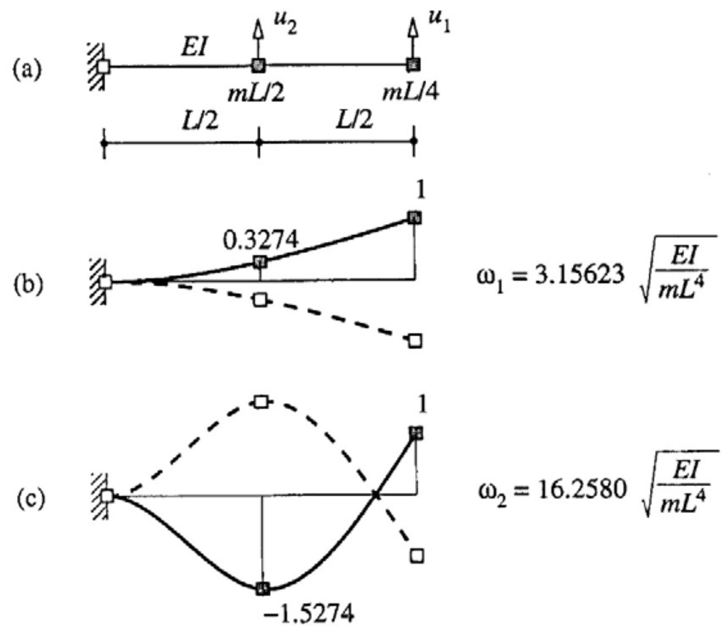


FIGURE: 2-DOF system: Rigid bar on elastic supports.



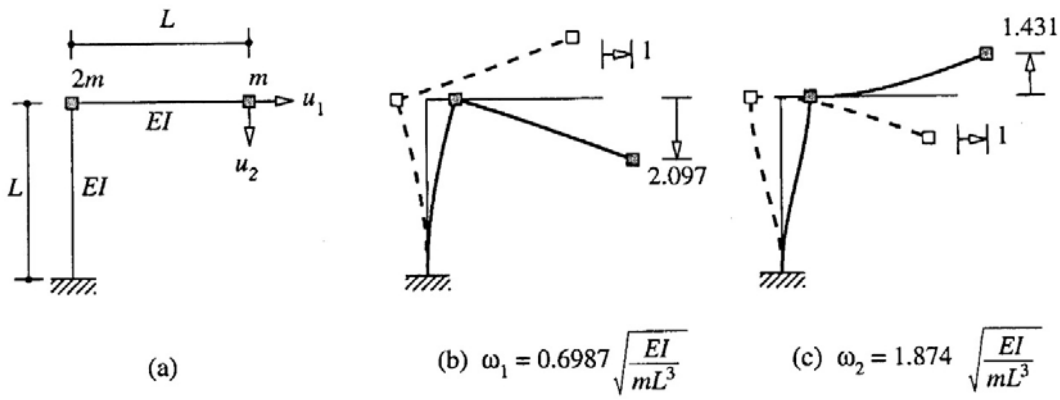


FIGURE: Cantilever beam & L-shaped frame modeled as 2-DOF systems.

NOTES:

Kinetic and Strain Energy

Consider the discretized model of a structure. Its **strain energy** is given by $V = \left(\frac{1}{2}\right) \mathbf{u}^T \mathbf{k} \mathbf{u}$, while its **kinetic energy** $T = \left(\frac{1}{2}\right) \dot{\mathbf{u}}^T \mathbf{m} \dot{\mathbf{u}}$.

NOTE: The above expressions for the kinetic and strain energies of a discretized structure are justified/obtained as follows:

The forces that are acting on the masses of the structure along the DOFs and accelerate them are equal to: $\frac{d}{dt}(\mathbf{m}\dot{\mathbf{u}})$. The work produced by moving the masses of the structure from state/position 1 to state/position 2 is equal to: $W_{12}^{(T)} = \int_1^2 \left[\frac{d}{dt}(\mathbf{m}\dot{\mathbf{u}})\right]^T d\mathbf{u} = \int_1^2 \left[\frac{d}{dt}(\mathbf{m}\dot{\mathbf{u}})\right]^T \dot{\mathbf{u}} dt = \int_1^2 d\left(\frac{1}{2}\dot{\mathbf{u}}^T \mathbf{m}\dot{\mathbf{u}}\right) = T_2 - T_1$, where $T_i \; (i = 1,2)$ is the kinetic energy at state/position i.

In an analogous fashion, the strain energy of the structure may be introduced in terms of the work performed by the elastic forces $\mathbf{f}_S = \mathbf{k}\mathbf{u}$ as the structure is deformed from state/position 1 to state/position 2: $W_{12}^{(V)} = \int_1^2 \mathbf{f}_S^T d\mathbf{u} = \int_1^2 (\mathbf{k}\mathbf{u})^T d\mathbf{u} = \int_1^2 d\left(\left(\frac{1}{2}\right)\mathbf{u}^T \mathbf{k}\mathbf{u}\right) = V_2 - V_1$.

Both energy expressions presented above involve a **scalar function** of the form $f = \mathbf{x}^T \mathbf{A} \mathbf{x}$, where the matrix $\mathbf{A}$ (i.e., either $\mathbf{k}$ or $\mathbf{m}$) is a **real symmetric** matrix. The scalar function $f = \mathbf{x}^T \mathbf{A} \mathbf{x}$ is known as a ***quadratic form***. If **for any real vector** $\mathbf{x} \neq \mathbf{0}$ the value of f is **positive**, *i.e.*, $f = \mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \;\; \forall \, \mathbf{x} \neq \mathbf{0} \in \mathbb{R}^N$, then the quadratic form $f = \mathbf{x}^T \mathbf{A} \mathbf{x}$ is called ***positive definite*** and the associated matrix $\mathbf{A}$ is referred to as a ***positive definite matrix***. We have already argued that **for civil engineering structures both matrices $\mathbf{m}$ & $\mathbf{k}$** are **positive definite** (because **civil engineering structures are stable and rigidly attached to the ground** and, additionally, **all DOFs associated with zero mass/inertia have been eliminated by static condensation**). Simply stated, for any deformed shape $\mathbf{u}$, the strain energy of the structure is positive, i.e., $V = \left(\frac{1}{2}\right) \mathbf{u}^T \mathbf{k} \mathbf{u} > 0$. Similarly, for any velocity vector $\dot{\mathbf{u}}$, the kinetic energy of the structure is positive, i.e., $T = \left(\frac{1}{2}\right) \dot{\mathbf{u}}^T \mathbf{m} \dot{\mathbf{u}} > 0$.

Linear Dependence

- We consider a set of **real vectors** $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$ in **linear space** $\mathcal{L}$ and a set of **real scalars** $\alpha_1, \alpha_2, \cdots, \alpha_S$. Then, the vector $\mathbf{x}$ given by:
$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_S \mathbf{x}_S$$
is said to be a **linear combination** of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$ with coefficients $\alpha_1, \alpha_2, \cdots, \alpha_S$.
- The vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$ are said to be ***linearly independent*** if the relation:
$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_S \mathbf{x}_S = \mathbf{0} \qquad (1)$$

can be satisfied only for the trivial case, *i.e.*, only when **all** the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_S$ are **identically zero**.

- If relation (1) is satisfied and at least one of the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_S$ is different from zero, then the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$ are said to be ***linearly dependent***, with the implication that **one vector is linear combination of the remaining $(S-1)$ vectors**.
  Thus, considering **vectors in the 3D space**, **two (non-zero) vectors** are **linearly dependent** if they are **collinear** (*i.e.*, parallel), and **three (non-zero) vectors** are **linearly dependent** if they are **coplanar** (*i.e.*, lie in the same plane).
- The subspace $\boldsymbol{\mathcal{S}}$ of $\boldsymbol{\mathcal{L}}$ consisting of the linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$ is called a subspace ***spanned*** by the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$.

  If $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{L}}$, then $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_S$ are said to ***span*** $\boldsymbol{\mathcal{L}}$.

How is the concept of *'Linear Independence'* relevant to our discussion?

The modal shapes $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_N$ of a civil engineering structure are linearly independent and, consequently, **any deformed shape $\mathbf{u}$ of the structure may be expressed as a linear combination of the modal shapes** (***modal expansion theorem***). Evidently, the modal shapes $\boldsymbol{\phi}_n$ $(n = 1,2,\cdots,N)$ play the same role as the three ***basis vectors*** $\boldsymbol{i} = [1,0,0]^T$, $\boldsymbol{j} = [0,1,0]^T$, and $\boldsymbol{k} = [0,0,1]^T$, in the 3D space. Any vector $\mathbf{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$ in the 3D space may be expressed as a linear combination of the above three basis vectors, or in matrix form

$$\mathbf{v} \;=\; v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} \;=\; v_x\begin{Bmatrix}1\\0\\0\end{Bmatrix} + v_y\begin{Bmatrix}0\\1\\0\end{Bmatrix} + v_z\begin{Bmatrix}0\\0\\1\end{Bmatrix} \;=\; \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}v_x\\v_y\\v_z\end{Bmatrix} \;=\; \begin{Bmatrix}v_x\\v_y\\v_z\end{Bmatrix}$$

The interested reader may consult the following two references:

HILDEBRAND, F.B. (1965), *Methods of Applied Mathematics*, 2nd Edition, PRENTICE-HALL, Inc.
[Chapter 1: Matrices and Linear Equations]

STRANG, G. (1980). *Linear Algebra and Its Applications*, 2nd Edition, ACADEMIC PRESS, Inc.

Since $\mathbf{k}$ & $\mathbf{m}$ are **both** **positive definite**, it follows that all $\lambda_j \ (j = 1,2,\cdots,N)$ are **positive**. Then, it is convenient to introduce the notation:

$$\boxed{\lambda_j = \omega_j^2 \quad (j = 1,2,\cdots,N)}$$

where $\omega_j \ (j = 1,2,\cdots,N)$ are **real positive numbers**.

Recall that $\lambda = -s^2$. Therefore, $s = \pm\sqrt{-\lambda} = \pm\sqrt{-\omega^2} = \pm i\omega$ we conclude that to each eigenvalue λ_j there corresponds the **pair of pure imaginary complex conjugate exponents**:

$$\left.\begin{matrix} s_j \\ \bar{s}_j \end{matrix}\right\} = \pm i\omega_j \quad (j = 1,2,\cdots,N)$$

Introducing the exponents into the expression $\mathbf{u}(t) = e^{st}\boldsymbol{\phi}$, **we conclude that the equation of motion** $\mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{0}$ **admits synchronous solutions of the form**:

$$\begin{aligned} \mathbf{u}_j(t) &= \left(C_j e^{i\omega_j t} + \bar{C}_j e^{-i\omega_j t}\right)\boldsymbol{\phi}_j \\ &= \underbrace{\left[A_j \cos(\omega_j t) + B_j \sin(\omega_j t)\right]}_{q_j(t)} \boldsymbol{\phi}_j \\ &= \underbrace{\rho_j \cos(\omega_j t - \theta_j)}_{q_j(t)} \boldsymbol{\phi}_j \qquad (j = 1,2,\cdots,N) \\ &= \underbrace{\rho_j \sin(\omega_j t + \vartheta_j)}_{q_j(t)} \boldsymbol{\phi}_j \end{aligned}$$

where: ρ_j = **amplitude**; $\rho_j = \sqrt{A_j^2 + B_j^2}$

θ_j = **phase angle**; $\theta_j = \tan^{-1}(B_j/A_j)$ [Consult the notes on SDOF System]

ϑ_j = **phase angle**; $\vartheta_j = \tan^{-1}(A_j/B_j)$ [Consult the notes on SDOF System]

ω_j = **natural frequency** corresponding to **natural mode of vibration** $\boldsymbol{\phi}_j$

($\boldsymbol{\phi}_j$ are also referred to as ***eigenvectors***, or ***characteristic vectors***, or ***natural modes***)

It is straightforward to demonstrate that $\mathbf{u}_j(t) = q_j(t)\boldsymbol{\phi}_j$ **satisfies the equation of motion** $\mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{0}$. Indeed, **by direct substitution**, $\left[-\omega_j^2 \mathbf{m}\boldsymbol{\phi}_j + \mathbf{k}\boldsymbol{\phi}_j\right] q_j(t) = \mathbf{0} \Leftrightarrow \mathbf{k}\boldsymbol{\phi}_j = \omega_j^2 \mathbf{m}\boldsymbol{\phi}_j$.

Clearly, the **synchronous motions** that we have established are **harmonic** (i.e., all DOFs **oscillate harmonically** about their equilibrium position **with the same circular frequency**).

FREE VIBRATION RESPONSE – UNDAMPED SYSTEM

We want to solve the **Equation of Motion** $\mathbf{m\ddot{u}} + \mathbf{ku} = \mathbf{0}$, subject to the following initial conditions: $\mathbf{u}(t=0) = \mathbf{u}_0$ & $\dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_0$.

We express the solution/response by invoking the **modal expansion theorem**:

$$\mathbf{u}(t) = \sum_{n=1}^{N} (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))\boldsymbol{\phi}_n = \sum_{n=1}^{N} q_n(t)\boldsymbol{\phi}_n$$

$$\dot{\mathbf{u}}(t) = \sum_{n=1}^{N} (-A_n \sin(\omega_n t) + B_n \cos(\omega_n t))\omega_n\boldsymbol{\phi}_n = \sum_{n=1}^{N} \dot{q}_n(t)\boldsymbol{\phi}_n$$

where N is the **total number of DOF** of the model of the discretized structure.

Therefore,

$$\mathbf{u}_0 = \mathbf{u}(0) = \sum_{r=1}^{N} A_r\boldsymbol{\phi}_r = \sum_{r=1}^{N} q_r(0)\boldsymbol{\phi}_r$$

$$\dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0) = \sum_{r=1}^{N} B_r\omega_r\boldsymbol{\phi}_r = \sum_{r=1}^{N} \dot{q}_r(0)\boldsymbol{\phi}_r$$

Pre-multiplying, both equations, by $\boldsymbol{\phi}_n^T\mathbf{m}$ and invoking the **eigenvector orthogonality theorem** we obtain:

$$\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0 = A_n\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n \quad\Rightarrow\quad A_n = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0}{\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_r} = q_n(0)$$

$$\boldsymbol{\phi}_n^T\mathbf{m}\dot{\mathbf{u}}_0 = B_n\omega_n\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n \quad\Rightarrow\quad B_n\omega_n = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0}{\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n} = \dot{q}_n(0)$$

Therefore, the free vibration response of the undamped system is expressed as follows:

$$\boxed{\begin{array}{c} \mathbf{u}(t) = \sum_{n=1}^{N} q_n(t)\boldsymbol{\phi}_n = \sum_{n=1}^{N}\left(q_n(0)\cos(\omega_n t) + \frac{\dot{q}_n(0)}{\omega_n}\sin(\omega_n t)\right)\boldsymbol{\phi}_n \\ q_n(0) = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0}{\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n} = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\mathbf{u}_0}{M_n} \quad , \quad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\dot{\mathbf{u}}_0}{\boldsymbol{\phi}_n^T\mathbf{m}\boldsymbol{\phi}_n} = \frac{\boldsymbol{\phi}_n^T\mathbf{m}\dot{\mathbf{u}}_0}{M_n} \end{array}}$$

IMPORTANT NOTE: Evidently, **for the structure to vibrate only in the n-th mode**, the initial displacement $\mathbf{u}_0$ and/or the initial velocity $\dot{\mathbf{u}}_0$ **must both be proportional to** $\boldsymbol{\phi}_n$, i.e., $\mathbf{u}_0 \sim \boldsymbol{\phi}_n$ & $\dot{\mathbf{u}}_0 \sim \boldsymbol{\phi}_n$.

FREE VIBRATION RESPONSE –SYSTEM WITH DAMPING

Statement of the problem:

$$\begin{array}{|ll|} \hline \textit{Equation of Motion:} & \mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = \mathbf{0} \\ \textit{Initial Conditions:} & \mathbf{u}_0 = \mathbf{u}(0) \quad , \quad \dot{\mathbf{u}}_0 = \dot{\mathbf{u}}(0) \\ \hline \end{array}$$

We are seeking a solution to the above problem. Procedures to obtain the desired solution differ depending on the form of damping: ***classical*** or ***non-classical***; these terms are defined next.

For **civil engineering structures** it is reasonable to assume that the matrices $\mathbf{m}$, $\mathbf{k}$, & $\mathbf{c}$ are all ***symmetric*** and ***positive definite***.

It can be demonstrated that the ***necessary*** and ***sufficient*** condition for the above system to possess natural modes of vibration, that are **real-valued and identical to those of the associated undamped system** (also referred to as ***classical normal modes***), is the following equality:

$$\boxed{\mathbf{c}\mathbf{m}^{-1}\mathbf{k} = \mathbf{k}\mathbf{m}^{-1}\mathbf{c}}$$

or, **equivalently**, that $(\mathbf{m}^{-1}\mathbf{k})$ & $(\mathbf{m}^{-1}\mathbf{c})$ commute, i.e., $(\mathbf{m}^{-1}\mathbf{k})(\mathbf{m}^{-1}\mathbf{c}) = (\mathbf{m}^{-1}\mathbf{c})(\mathbf{m}^{-1}\mathbf{k})$.

Let $\mathbf{\Phi}$ be the ***modal matrix*** and $\mathbf{\Omega}^2$ the ***spectral matrix***. Specifically,

$$\mathbf{\Phi} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n \quad \cdots \quad \boldsymbol{\phi}_N] \quad \mathbf{\Omega}^2 = \begin{bmatrix} \omega_1^2 & & & \\ & \omega_2^2 & & \\ & & \ddots & \\ & & & \omega_N^2 \end{bmatrix}$$

Invoking the **eigenvector expansion theorem**, we may express the structural response as $\mathbf{u}(t) = \mathbf{\Phi q}(t)$. Then,

$$\mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = \mathbf{0} \quad \Rightarrow \quad \mathbf{m\Phi\ddot{q}}(t) + \mathbf{c\Phi\dot{q}}(t) + \mathbf{k\Phi q}(t) = \mathbf{0} \quad \overset{\mathbf{\Phi}^T}{\Rightarrow} \quad \mathbf{m\Phi\ddot{q}}(t) + \mathbf{c\Phi\dot{q}}(t) + \mathbf{k\Phi q}(t) = \mathbf{0}$$

$$\mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = \mathbf{0}$$
$$\Downarrow$$
$$\mathbf{m\Phi\ddot{q}}(t) + \mathbf{c\Phi\dot{q}}(t) + \mathbf{k\Phi q}(t) = \mathbf{0}$$
$$\Downarrow$$
$$\mathbf{\Phi}^T| \quad \mathbf{m\Phi\ddot{q}}(t) + \mathbf{c\Phi\dot{q}}(t) + \mathbf{k\Phi q}(t) = \mathbf{0}$$
$$\Downarrow$$
$$\underbrace{\mathbf{\Phi}^T\mathbf{m\Phi}}_{\mathbf{M}}\,\mathbf{\ddot{q}}(t) + \underbrace{\mathbf{\Phi}^T\mathbf{c\Phi}}_{\mathbf{C}}\,\mathbf{\dot{q}}(t) + \underbrace{\mathbf{\Phi}^T\mathbf{k\Phi}}_{\mathbf{K}}\,\mathbf{q}(t) = \mathbf{0}$$
$$\Downarrow$$
$$\mathbf{M\ddot{q}}(t) + \mathbf{C\dot{q}}(t) + \mathbf{Kq}(t) = \mathbf{0}$$

where

$$\mathbf{M} = \mathbf{\Phi}^T\mathbf{m\Phi} = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_N \end{bmatrix} \quad \mathbf{C} = \mathbf{\Phi}^T\mathbf{c\Phi} = \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_N \end{bmatrix} \quad \mathbf{K} = \mathbf{\Phi}^T\mathbf{k\Phi} = \begin{bmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_N \end{bmatrix}$$

$$M_n = \boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_n \quad , \quad C_n = \boldsymbol{\phi}_n^T \mathbf{c} \boldsymbol{\phi}_n \quad , \quad K_n = \boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_n$$

Therefore

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = 0$$

At this point we introduce the **damping ration of each mode**:

$$\xi_n = \frac{C_n}{2 M_n \omega_n}$$

Therefore

$$\boxed{\ddot{q}_n + 2\xi_n \omega_n \dot{q}_n + \omega_n^2 q_n = 0 \quad (n = 1,2,\cdots,N)}$$

Noticing that the above equation is mathematically identical to the equation that governs the free vibration response of a SDOF with damping, we may write

$$q_n(t) = e^{-\xi_n \omega_n t}\left[q_n(0)\cos(\omega_{Dn} t) + \frac{\dot{q}_n(0) + \xi_n \omega_n q_n(0)}{\omega_{Dn}} \sin(\omega_{Dn} t)\right]$$

$$\text{where:} \quad \omega_{Dn} = \omega_n \sqrt{1 - \xi_n^2} \quad , \quad q_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \mathbf{u}_0}{M_n} \quad , \quad \dot{q}_n(0) = \frac{\boldsymbol{\phi}_n^T \mathbf{m} \dot{\mathbf{u}}_0}{M_n}$$

Recall that

$$\boxed{\mathbf{u}(t) = \boldsymbol{\Phi} \mathbf{q}(t) = \sum_{n=1}^{N} q_n(t) \boldsymbol{\phi}_n}$$

IMPORTANT NOTE: As for the undamped case, **for the structure to vibrate only in the n-th mode**, the initial displacement $\mathbf{u}_0$ and/or the initial velocity $\dot{\mathbf{u}}_0$ **must both be proportional to** $\boldsymbol{\phi}_n$, i.e., $\mathbf{u}_0 \sim \boldsymbol{\phi}_n$ & $\dot{\mathbf{u}}_0 \sim \boldsymbol{\phi}_n$.

APPENDIX

The *properties of the Generalized Characteristic-Value Problem* $\mathbf{ku} = \lambda\mathbf{mu}$ that we enumerated earlier may be demonstrated by following one of the following two strategies:

(1) One may **work directly with** the *Generalized Characteristic-Value Problem* $\mathbf{ku} = \lambda\mathbf{mu}$, or
(2) One may convert/transform the problem $\mathbf{ku} = \lambda\mathbf{mu}$ to an *equivalent Standard Eigenvalue Problem* of the form $\mathbf{Av} = \lambda\mathbf{v}$, where the matrix $\mathbf{A}$ is symmetric and positive definite.

Let us start by considering **strategy #1**:

We start by demonstrating that the characteristic values ($\lambda_r = \omega_r^2$) and modal shapes ($\boldsymbol{\phi}_r$) of a civil engineering structure are real.

Theorem

The **eigenvalues** λ of a Generalized Eigenvalue Problem $\mathbf{ku} = \lambda\mathbf{mu}$, with $\mathbf{k}$ & $\mathbf{m}$ **real & symmetric matrices**, and $\mathbf{m}$ is **positive definite**, are **real**.

As a corollary, **the corresponding eigenvectors are also real**.

Proof

We consider the eigenvalue-eigenvector pair $(\lambda_r, \boldsymbol{\phi}_r)$ and assume they are complex.

Because $\mathbf{k}$ & $\mathbf{m}$ are **real**, it follows that the complex conjugate pair $(\bar{\lambda}_r, \bar{\boldsymbol{\phi}}_r)$ must also constitute an eigenvalue-eigenvector pair.

Therefore:

$$\left.\begin{aligned}\mathbf{k}\boldsymbol{\phi}_r &= \lambda_r\mathbf{m}\boldsymbol{\phi}_r\\ \mathbf{k}\bar{\boldsymbol{\phi}}_r &= \bar{\lambda}_r\mathbf{m}\bar{\boldsymbol{\phi}}_r\end{aligned}\right\}\Rightarrow$$

$$\left.\begin{aligned}\bar{\boldsymbol{\phi}}_r^T\mathbf{k}\boldsymbol{\phi}_r &= \lambda_r\bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r\\ (\mathbf{k}\bar{\boldsymbol{\phi}}_r)^T\boldsymbol{\phi}_r &= \bar{\lambda}_r(\mathbf{m}\bar{\boldsymbol{\phi}}_r)^T\boldsymbol{\phi}_r\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\bar{\boldsymbol{\phi}}_r^T\mathbf{k}\boldsymbol{\phi}_r &= \lambda_r\bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r\\ \bar{\boldsymbol{\phi}}_r^T\mathbf{k}^T\boldsymbol{\phi}_r &= \bar{\lambda}_r\bar{\boldsymbol{\phi}}_r^T\mathbf{m}^T\boldsymbol{\phi}_r\end{aligned}\right\} \overset{\textit{Recall}}{\underset{\substack{\mathbf{m}=\mathbf{m}^T\\ \mathbf{k}=\mathbf{k}^T}}{\Rightarrow}}$$

$$\underbrace{\bar{\boldsymbol{\phi}}_r^T\mathbf{k}\boldsymbol{\phi}_r - \bar{\boldsymbol{\phi}}_r^T\mathbf{k}^T\boldsymbol{\phi}_r}_{0} = (\lambda_r - \bar{\lambda}_r)\bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r$$

Recalling that the **mass matrix** $\mathbf{m}$ is a **real**, **symmetric**, and **positive definite** matrix, it is straight forward to demonstrate that $\bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r$ is real and positive. Indeed,

$$\left(\overline{\bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r}\right) = \boldsymbol{\phi}_r^T\mathbf{m}\bar{\boldsymbol{\phi}}_r = \boldsymbol{\phi}_r^T\mathbf{m}^T\bar{\boldsymbol{\phi}}_r = (\mathbf{m}\boldsymbol{\phi}_r)^T\bar{\boldsymbol{\phi}}_r = ((\mathbf{m}\boldsymbol{\phi}_r)^T\bar{\boldsymbol{\phi}}_r)^T = \bar{\boldsymbol{\phi}}_r^T\mathbf{m}^T\boldsymbol{\phi}_r = \bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r$$

Therefore, $\bar{\boldsymbol{\phi}}_r^T\mathbf{m}\boldsymbol{\phi}_r$ is **real** and **positive**.
It follows that

$$(\lambda_r - \bar{\lambda}_r) = 0$$

Which can be satisfied if and only if λ_r is **real**.

The eigenvectors also can be taken to be real, by rejecting permissible complex multiplicative factors.

Then we proceed to prove that modal shapes $\boldsymbol{\phi}_r$ & $\boldsymbol{\phi}_s$, corresponding to distinct eigenvalues, λ_r and λ_s, are mutually orthogonal relative to $\mathbf{m}$ and $\mathbf{k}$.

Theorem

If $\boldsymbol{\phi}_r$ & $\boldsymbol{\phi}_s$ are eigenvectors, corresponding to two distinct eigenvalues λ_r and λ_s, respectively, of the eigenvalue problem $(\mathbf{k}-\boldsymbol{\lambda}\mathbf{m})\boldsymbol{\phi} = \mathbf{0}$, where $\mathbf{k}$ & $\mathbf{m}$ are real and symmetric matrices, and $\mathbf{m}$ is positive definite, there follows:

$$\boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s = \mathbf{0} \quad , \quad \boldsymbol{\phi}_r^T\mathbf{k}\boldsymbol{\phi}_s = \mathbf{0} \quad , \quad \lambda_r \neq \lambda_s$$

We say that eigenvectors $\boldsymbol{\phi}_r$ & $\boldsymbol{\phi}_s$ are orthogonal (to each other) relative to $\mathbf{k}$ & $\mathbf{m}$.

Proof

If λ_r& λ_s are **distinct eigenvalues** corresponding, respectively, to the eigenvectors $\boldsymbol{\phi}_r$& $\boldsymbol{\phi}_s$, there follows:

$$\mathbf{k}\boldsymbol{\phi}_r = \lambda_r\mathbf{m}\boldsymbol{\phi}_r \quad , \quad \mathbf{k}\boldsymbol{\phi}_s = \lambda_s\mathbf{m}\boldsymbol{\phi}_s$$

and hence, also:

$$(\mathbf{k}\boldsymbol{\phi}_r)^T\boldsymbol{\phi}_s = \lambda_r(\mathbf{m}\boldsymbol{\phi}_r)^T\boldsymbol{\phi}_s \quad , \quad \boldsymbol{\phi}_r^T\mathbf{k}\boldsymbol{\phi}_s = \lambda_s\boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s$$

or, making use of the symmetry in $\mathbf{k}$ & $\mathbf{m}$,

$$\boldsymbol{\phi}_r^T\mathbf{k}\boldsymbol{\phi}_s = \lambda_r\boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s \qquad (1)$$

$$\boldsymbol{\phi}_r^T\mathbf{k}\boldsymbol{\phi}_s = \lambda_s\boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s \qquad (2)$$

$$(2) - (1) \quad \Rightarrow \quad (\lambda_s - \lambda_r)\boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s = 0$$

Thus, since $\lambda_s \neq \lambda_r$ by assumption, we conclude that:

$$\left.\begin{aligned} \boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s &= 0 \\ \left(\frac{1}{\lambda_s}\right)\boldsymbol{\phi}_r^T\mathbf{k}\boldsymbol{\phi}_s &= \boldsymbol{\phi}_r^T\mathbf{m}\boldsymbol{\phi}_s \end{aligned}\right\} \quad \Rightarrow \quad \boldsymbol{\phi}_r^T\mathbf{k}\boldsymbol{\phi}_s = 0$$

Finally, we demonstrate that the modal shapes $\boldsymbol{\phi}_r$ are linearly independent, a fact that forms the basis of the *modal superposition method.*

Theorem

For mass matrix $\mathbf{m}$ real, symmetric, and positive definite, any set of non-zero real vectors which are mutually orthogonal relative to $\mathbf{m}$ is a set of linearly independent vectors.

Proof

The above Theorem may easily be demonstrated **by contradiction**. Indeed, let us assume that $\mathbf{x}_i$, $(i = 1,2,\cdots,\ell,\cdots,S)$ are S vectors that are mutually orthogonal relative to $\mathbf{m}$ and **let us assume they are linearly dependent**, *i.e.*, $\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_\ell\mathbf{x}_\ell + \cdots + \alpha_S\mathbf{x}_S = \mathbf{0} \Rightarrow \mathbf{x}_\ell = -\sum_{i\neq\ell}(\alpha_i/\alpha_\ell)\mathbf{x}_i$. We pre-multiply by $\mathbf{x}_\ell^T\mathbf{m}$ and we exploit the orthogonality property: $\mathbf{x}_\ell^T\mathbf{m}\mathbf{x}_\ell = -\sum_{i\neq\ell}(\alpha_i/\alpha_\ell)\mathbf{x}_\ell^T\mathbf{m}\mathbf{x}_i = -\sum_{i\neq\ell}0 = 0$. But this is a contradiction given that $\mathbf{m}$ is positive definite and, consequently, $\mathbf{x}_\ell^T\mathbf{m}\mathbf{x}_\ell > 0$. Thus, by contradiction, the mutually orthogonal, relative to $\mathbf{m}$, vectors $\mathbf{x}_i$, $(i = 1,2,\cdots,\ell,\cdots,S)$ are **linearly independent**.

Now let us consider **strategy #2**:

A quick way to transform the *generalized eigenvalue problem* $\mathbf{ku} = \lambda\mathbf{mu}$ to the *standard eigenvalue problem* of the form $\mathbf{Av} = \lambda\mathbf{v}$, is the following: $\mathbf{ku} = \lambda\mathbf{mu} \Leftrightarrow (\mathbf{m}^{-1}\mathbf{k})\mathbf{u} = \lambda\mathbf{u}$. However, the matrix $(\mathbf{m}^{-1}\mathbf{k})$ is **not symmetric**, and this obscures the problem. For this reason, we adopt a different approach.

We start with the following theorem:

Theorem

If matrix $\mathbf{B}$ is **real, symmetric** and **positive definite**, then **there exists a non-singular real matrix $\mathbf{W}$** such that $\mathbf{B} = \mathbf{W}^T\mathbf{W}$

Proof

The matrix $\mathbf{B}$ can be written as:

$$\mathbf{B} = \mathbf{R}\mathbf{D}\mathbf{R}^T$$

where:

- the **columns** of matrix $\mathbf{R}$ consist of the ***orthonormal eigenvectors*** of $\mathbf{B}$, and
- $\mathbf{D} = \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_N)$, where the numbers $\mu_i > 0$ $(i = 1,2,\cdots,N)$, are the corresponding eigenvalues of $\mathbf{B}$.

Indeed, let $\mathbf{e}_r$ be the orthonormal eigenvectors of $\mathbf{B}$, *i.e.*, $\mathbf{B}\mathbf{e}_r = \mu_r\mathbf{e}_r$. Then,

$$\mathbf{R} = \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix}$$

Evidently

$$\mathbf{R}^T\mathbf{R} = \begin{pmatrix} \rightarrow & \mathbf{e}_1^T & \rightarrow \\ \rightarrow & \mathbf{e}_2^T & \rightarrow \\ \vdots & \vdots & \vdots \\ \rightarrow & \mathbf{e}_N^T & \rightarrow \end{pmatrix} \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} = \mathbf{I}$$

Therefore, $\mathbf{R}^T\mathbf{R} = \mathbf{I} \Leftrightarrow \mathbf{R}^T = \mathbf{R}^{-1} \Leftrightarrow (\mathbf{R}^T)^{-1} = \mathbf{R}$.

We may write

$$\mathbf{BR} = \mathbf{B}\begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} = \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \\ \mu_1\mathbf{e}_1 & \mu_2\mathbf{e}_2 & \cdots & \mu_N\mathbf{e}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} = \mathbf{RD}$$

It follows that: $\mathbf{BR} = \mathbf{RD} \iff \mathbf{BRR}^T = \mathbf{RDR}^T \iff \mathbf{BI} = \mathbf{RDR}^T \iff \mathbf{B} = \mathbf{RDR}^T$

Then:

$$\mathbf{B} = \mathbf{RDR}^T = \mathbf{R}\sqrt{\mathbf{D}}\sqrt{\mathbf{D}}\mathbf{R}^T = \left(\sqrt{\mathbf{D}}\mathbf{R}^T\right)^T\left(\sqrt{\mathbf{D}}\mathbf{R}^T\right) = \mathbf{W}^T\mathbf{W}$$

where: $\boxed{\mathbf{W} = \sqrt{\mathbf{D}}\mathbf{R}^T}$

Matrices, $\mathbf{R}^T$ & $\mathbf{R}$**,** are **invertible** because they are **non-singular** (**linear independence of eigenvectors**).
Consequently, also **W** is **non-singular**.
Notice that:

$$(\mathbf{W}^T)^{-1} = \left(\mathbf{R}\sqrt{\mathbf{D}}\right)^{-1} = \left(\sqrt{\mathbf{D}}\right)^{-1}\mathbf{R}^{-1}$$

$$(\mathbf{W}^{-1})^T = \left[(\mathbf{R}^T)^{-1}\left(\sqrt{\mathbf{D}}\right)^{-1}\right]^T = \left(\sqrt{\mathbf{D}}\right)^{-1}\mathbf{R}^T = \left(\sqrt{\mathbf{D}}\right)^{-1}\mathbf{R}^{-1}$$

Therefore:

$$\boxed{(\mathbf{W}^T)^{-1} = (\mathbf{W}^{-1})^T}$$

Using the above Theorem, the mass matrix **m**, being a real, symmetric and positive definite matrix, may be resolved as follows:

$$\mathbf{m} = \mathbf{Q}^T\mathbf{Q}$$

where **Q** is a **real**, **nonsingular matrix**.
Therefore:

$$\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi} \quad \Longrightarrow \quad \mathbf{k}\boldsymbol{\phi} = \lambda\left(\mathbf{Q}^T\mathbf{Q}\right)\boldsymbol{\phi} \qquad \textbf{(1)}$$

Next, we consider the ***linear transformation***:

$$\boxed{\mathbf{Q}\boldsymbol{\phi} = \mathbf{v}} \qquad \textbf{(2)}$$

from which we obtain the ***inverse transformation***:

$$\boldsymbol{\phi} = \mathbf{Q}^{-1}\mathbf{v} \qquad \textbf{(3)}$$

The inverse $\mathbf{Q}^{-1}$ is guaranteed to exist because **Q** is **non-singular**.
Introducing (2) & (3) into (1) and pre-multiplying on the left by $\left(\mathbf{Q}^T\right)^{-1}$, we obtain the eigenvalue problem:

$$\boxed{\mathbf{Av} = \lambda\mathbf{v}}$$

where, considering the relation $\left(\mathbf{Q}^T\right)^{-1} = \left(\mathbf{Q}^{-1}\right)^T$, we conclude that:

$$\mathbf{A} = \left(\mathbf{Q}^T\right)^{-1}\mathbf{k}\mathbf{Q}^{-1} = \left(\mathbf{Q}^{-1}\right)^T\mathbf{k}\mathbf{Q}^{-1} = \mathbf{A}^T$$

i.e., matrix $\mathbf{A} = \left(\mathbf{Q}^T\right)^{-1}\mathbf{k}\mathbf{Q}^{-1} = \left(\mathbf{Q}^{-1}\right)^T\mathbf{k}\mathbf{Q}^{-1}$ is a **real**, **symmetric matrix**.

It is evident that the original $\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi}$ has the **same eigenvalues** as the problem $\mathbf{Av} = \lambda\mathbf{v}$ (with $\mathbf{A} = \left(\mathbf{Q}^T\right)^{-1}\mathbf{k}\mathbf{Q}^{-1} = \left(\mathbf{Q}^{-1}\right)^T\mathbf{k}\mathbf{Q}^{-1}$) and the **eigenvectors are related as** follows: $\mathbf{Q}\boldsymbol{\phi} = \mathbf{v} \iff \boldsymbol{\phi} = \mathbf{Q}^{-1}\mathbf{v}$.

Furthermore, the orthogonality properties of the eigenvectors of one problem imply the orthogonality properties of the eigenvectors of the other problem. Specifically,

'**m**-*orthogonality*' $\boldsymbol{\phi}_i^T \mathbf{m} \boldsymbol{\phi}_j = \boldsymbol{\phi}_i^T \mathbf{Q}^T \mathbf{Q} \boldsymbol{\phi}_j = \mathbf{v}_i^T \mathbf{v}_j = 0 \quad (i \neq j)$, which leads to:

'**k**-*orthogonality*' $\boldsymbol{\phi}_i^T \mathbf{k} \boldsymbol{\phi}_j = \lambda_j \boldsymbol{\phi}_i^T \mathbf{m} \boldsymbol{\phi}_j = 0 \quad (i \neq j)$

The real eigenvalues $\lambda_r, (r = 1,2,\cdots,N)$ of the problem $\mathbf{k}\boldsymbol{\phi} = \lambda \mathbf{m} \boldsymbol{\phi}$ are all positive because $\boldsymbol{\phi}_r^T \mathbf{m} \boldsymbol{\phi}_r > 0$ and $\boldsymbol{\phi}_r^T \mathbf{k} \boldsymbol{\phi}_r > 0$. Specifically,

$$\mathbf{k}\boldsymbol{\phi}_r = \lambda_r \mathbf{m} \boldsymbol{\phi}_r \Leftrightarrow \boldsymbol{\phi}_r^T \mathbf{k} \boldsymbol{\phi}_r = \lambda_r \boldsymbol{\phi}_r^T \mathbf{m} \boldsymbol{\phi}_r \Leftrightarrow \lambda_r = (\boldsymbol{\phi}_r^T \mathbf{k} \boldsymbol{\phi}_r)/(\boldsymbol{\phi}_r^T \mathbf{m} \boldsymbol{\phi}_r) > 0$$

For completeness we remind the reader that the eigenvalues λ_r are also the eigenvalues of the problem $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ [with $\mathbf{A} = \left(\mathbf{Q}^{-1}\right)^T \mathbf{k} \mathbf{Q}^{-1}$], which implies that matrix **A**, besides being **real** and **symmetric**, is also **positive definite**.

<u>NOTE</u>: The relationship $\left(\mathbf{Q}^T\right)^{-1} = \left(\mathbf{Q}^{-1}\right)^T$ is obtained as follows:

$$\mathbf{I} = (\mathbf{I})^T \quad \Leftrightarrow \quad \mathbf{I} = \left(\mathbf{Q}^{-1}\mathbf{Q}\right)^T \quad \Leftrightarrow \quad \mathbf{I} = \mathbf{Q}^T\left(\mathbf{Q}^{-1}\right)^T \quad \Leftrightarrow \quad \left(\mathbf{Q}^T\right)^{-1} = \left(\mathbf{Q}^{-1}\right)^T$$

<u>NOTE</u>:

The ***orthogonality property*** of eigenvectors, demonstrated above, was based on the assumption that the corresponding **eigenvalues** are **<u>distinct</u>**.

The question arises as to **what happens when there are repeated eigenvalues**, *i.e.*, when two or more eigenvalues have the same value, and we note that **when an eigenvalue λ_i is repeated m_i times**, where m_i is integer, λ_i **is said to have multiplicity m_i**.

The answer to the above question lies in the following **Theorem**:

<u>Theorem</u>

If an eigenvalue λ_i of a <u>real</u>-<u>symmetric</u> matrix A has multiplicity m_i, then A <u>has exactly</u> m_i <u>linearly independent eigenvectors</u> corresponding to λ_i.

[For a proof see section **1.18** of HILDEBRAND, F.B. (1965), *Methods of Applied Mathematics*, 2nd Edition.]

These eigenvectors are **<u>not unique</u>**, as **any linear combination of the eigenvectors belonging to a repeated eigenvalue is also an eigenvector**.

The linearly independent eigenvectors corresponding to λ_i are **not necessarily orthogonal**. **<u>Any set of linearly independent eigenvectors, though, can be rendered orthogonal by a procedure known as the *Gram-Schmidt orthogonalization process*</u>**.

Of course, the eigenvectors belonging to the repeated eigenvalue are orthogonal to the eigenvectors belonging to the remaining eigenvalues. Hence, **<u>all the eigenvectors of a real symmetric matrix</u> A <u>are orthogonal regardless of whether there are repeated eigenvalues or not</u>**.