FOURIER TRANSFORM (FOURIER INTEGRAL)

Fourier Transform (Fourier Integral)	$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt$
Inverse Fourier Transform (Synthesis Equation)	$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{+i\omega t}\, d\omega$

The Fourier Integral does not converge for all functions.

The ***DIRICHLET conditions***,

- The function $f(t)$ is **absolutely integrable**, that is

$$\int_{-\infty}^{+\infty} |f(t)|\, dt < \infty$$

- $f(t)$ has a **finite number of maxima and minima** and a **finite number of discontinuities** in **any finite interval**,

provide a set of **sufficient conditions** for the **existence** of the **Fourier Transform** $F(\omega)$.

If $f(t)$ is absolutely integrable, then: $\lim_{\omega \to \pm\infty} F(\omega) = \lim_{\omega \to \pm\infty} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt = 0$

Intuitively this result derives from the fact that, **for large** ω, **the exponential oscillates faster than any length scale present in** $f(t)$. Thus, for ω large enough, $f(t)$ is essentially constant over each interval $2n\pi \leq \omega t \leq 2(n+1)\pi$ and the integral vanishes.

Functions that do not meet the *DIRICHLET* conditions may still have a Fourier Transform. These include **periodic functions**, whose transforms consist of impulses, and **functions whose Fourier Integral only converges as a limit**.

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REFERENCES:

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BRACEWELL, R.N. (1965). ***The Fourier Transform and its Applications***, **McGraw-Hill New York**

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We use the notation:

$$\boxed{f(t) \quad \leftrightarrow \quad F(\omega)}$$

to indicate that the functions $\boldsymbol{f(t)}$ & $\boldsymbol{F(t)}$ form a ***Fourier Transform pair***.

$$\boxed{F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt = R(\omega) + iX(\omega) = A(\omega) e^{i\phi(\omega)}}$$

where: $\boldsymbol{A(\omega)}$ = **Fourier Spectrum** of $\boldsymbol{f(t)}$

$\boldsymbol{A^2(\omega)}$ = **Energy Spectrum** of $\boldsymbol{f(t)}$

$\boldsymbol{\phi(\omega)}$ = **phase angle**

Real time functions:

If f is real, then the real and imaginary parts of $\boldsymbol{F(\omega) = R(\omega) + iX(\omega)}$ are given by:

$$R(\omega) = \int_{-\infty}^{+\infty} f(t) \cos(\omega t)\; dt \qquad X(\omega) = -\int_{-\infty}^{+\infty} f(t) \sin(\omega t)\; dt$$

From the above expressions we conclude that $\boldsymbol{R(\omega)}$ **is even** and $\boldsymbol{X(\omega)}$ **is odd**, *i.e.*

$$R(-\omega) = R(\omega) \qquad X(-\omega) = -X(\omega)$$

Therefore,

$$F(-\omega) = F^*(\omega) \quad (* = \textbf{\textit{complex conjugate}})$$

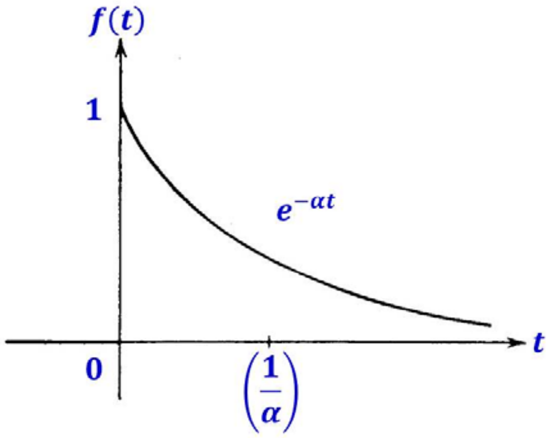
Conversely, if $\boldsymbol{F(-\omega) = F^*(\omega)}$ then $\boldsymbol{f(t)}$ = **real**.

Thus, $\boldsymbol{F(-\omega) = F^*(\omega)}$ is a necessary and sufficient condition for $\boldsymbol{f(t)}$ to be real, *i.e.*

$$\boxed{f(t) = real \quad \Leftrightarrow \quad F(-\omega) = F^*(\omega)}$$

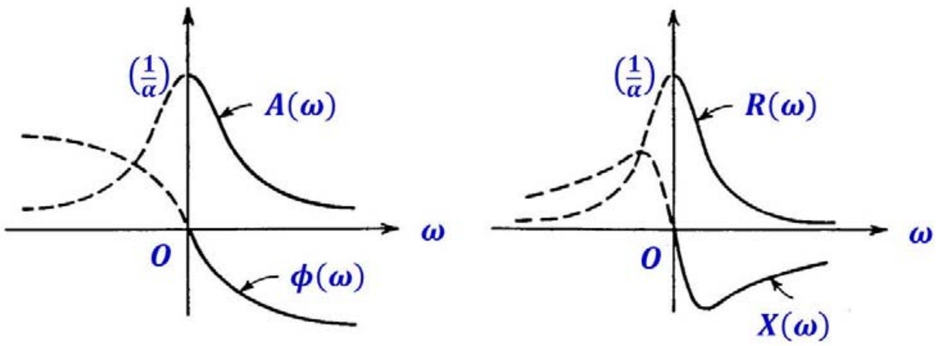
EXAMPLE: Let $\boldsymbol{f(t) = e^{-\alpha t}U(t)} \quad \boldsymbol{\alpha > 0}$, where: $\boldsymbol{U(t)}$ = ***Heaviside (unit step) function***.

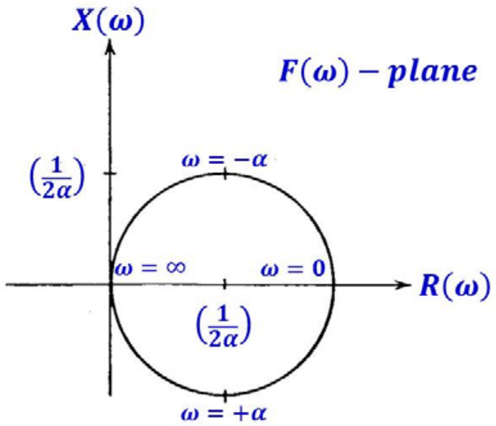
$$F(\omega) = \int_{-\infty}^{+\infty} e^{-\alpha t}U(t)e^{-i\omega t}\,dt = \int_{0}^{t} e^{-\alpha t}e^{-i\omega t}\,dt = \frac{1}{\alpha + i\omega}$$



In the FIGURE below, we show the various ways of plotting $\boldsymbol{F(\omega)}$:

$$F(\omega) = \frac{1}{\alpha + i\omega} = \frac{\alpha}{\alpha^2 + \omega^2} - i\,\frac{\omega}{\alpha^2 + \omega^2} = \frac{1}{\sqrt{\alpha^2 + \omega^2}}\,e^{-i\tan^{-1}(\omega/\alpha)}$$





EXAMPLE: Find the Fourier Transform of the ***signum function*** defined as:

$$f(t) = \operatorname{sgn}(t) = \begin{cases} -1 & t < 0 \\ +1 & t > 0 \end{cases}$$

which is **not absolutely integrable** and its Fourier integral does not converge.

We form the auxiliary function: $g(t) = \begin{cases} -e^{-\varepsilon t} & t < 0 \\ +e^{-\varepsilon t} & t > 0 \end{cases}$

which yields $f(t) = \operatorname{sgn}(t)$ as the limit: $f(t) = \lim_{\varepsilon \to 0} g(t)$

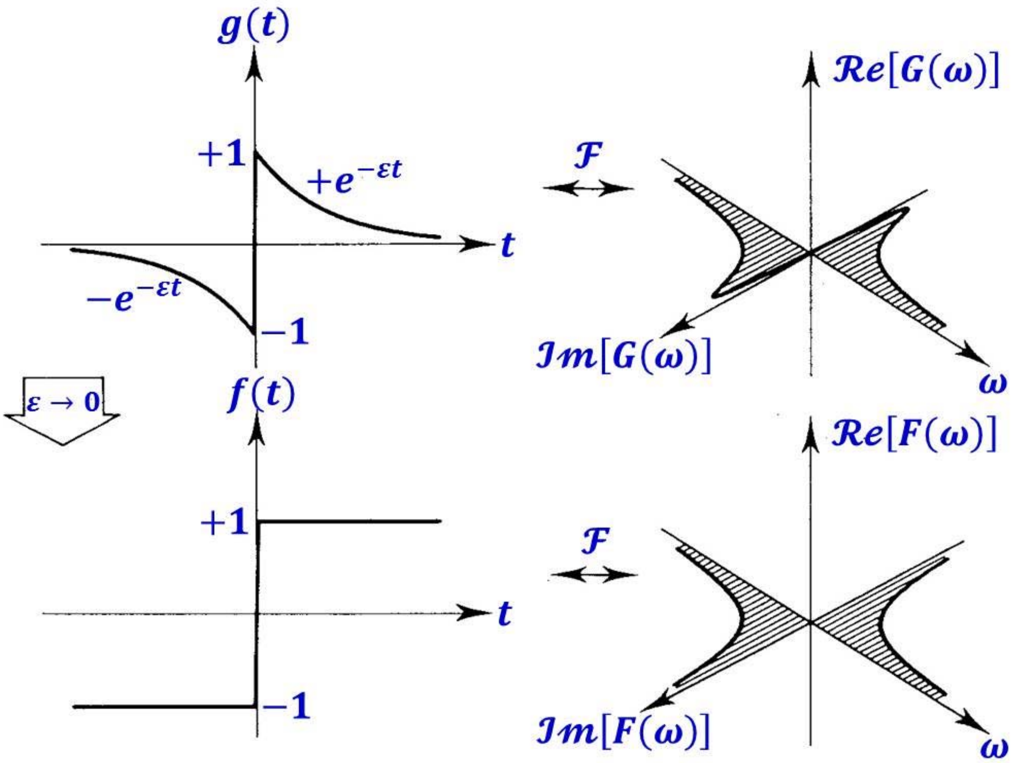
The Fourier transform of $g(t)$ is: $\mathcal{F}\{g(t)\} = \int_{-\infty}^{0} (-e^{\varepsilon t}) e^{-i\omega t}\, dt + \int_{0}^{+\infty} (+e^{\varepsilon t}) e^{-i\omega t}\, dt =$
$- \int_{-\infty}^{0} e^{(\varepsilon - i\omega)t}\, dt + \int_{0}^{+\infty} e^{-(\varepsilon + i\omega)t}\, dt = -\frac{1}{\varepsilon - i\omega} + \frac{1}{\varepsilon + i\omega}$

and the transform of $f(t)$ is obtained as the limit:

$$F(\omega) = \mathcal{F}\{\operatorname{sgn}(t)\} = \lim_{\varepsilon \to 0} \mathcal{F}\{g(t)\} = \frac{2}{i\omega} = -i\frac{2}{\omega}$$

This yields the Fourier transform pair: $\boxed{\operatorname{sgn}(t) \quad \leftrightarrow \quad -i\frac{2}{\omega}}$

which is **real odd in time**, hence **imaginary odd in frequency**.



Fourier Transform of signum function

NOTE: Some authors denote the Fourier transform of $f(t)$ by $F(i\omega)$ [instead of $F(\omega)$]. This is consistent with the fact that **the *Fourier Transform* is <u>a special case of</u> *Laplace Transform***.

SIMPLE THEOREMS:

The following is a list of simple theorems that can be easily derived from the Fourier integral and its inverse; it is assumed that all functions under consideration have Fourier integrals.

Most of these theorems are valid, in slightly modified forms, for the **Laplace Transform**, **the discreet classes of the Fourier Transform** and the **z-transform** [the ***z-transform*** is to discrete-time signals what the Laplace Transform is to their continuous-time counterparts].

LINEARITY:

Let: $f_1(t) \leftrightarrow F_1(\omega)$, $f_2(t) \leftrightarrow F_2(\omega)$, $\cdots$, $f_n(t) \leftrightarrow F_n(\omega)$

Then: $a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t) \quad \leftrightarrow \quad a_1 F_1(\omega) + a_2 F_2(\omega) + \cdots + a_n F_n(\omega)$

where: $a_1, a_2, \cdots, a_n$ are arbitrary constants.

Proof: The theorem is self-evident and is based on the linearity of the Fourier Integral.

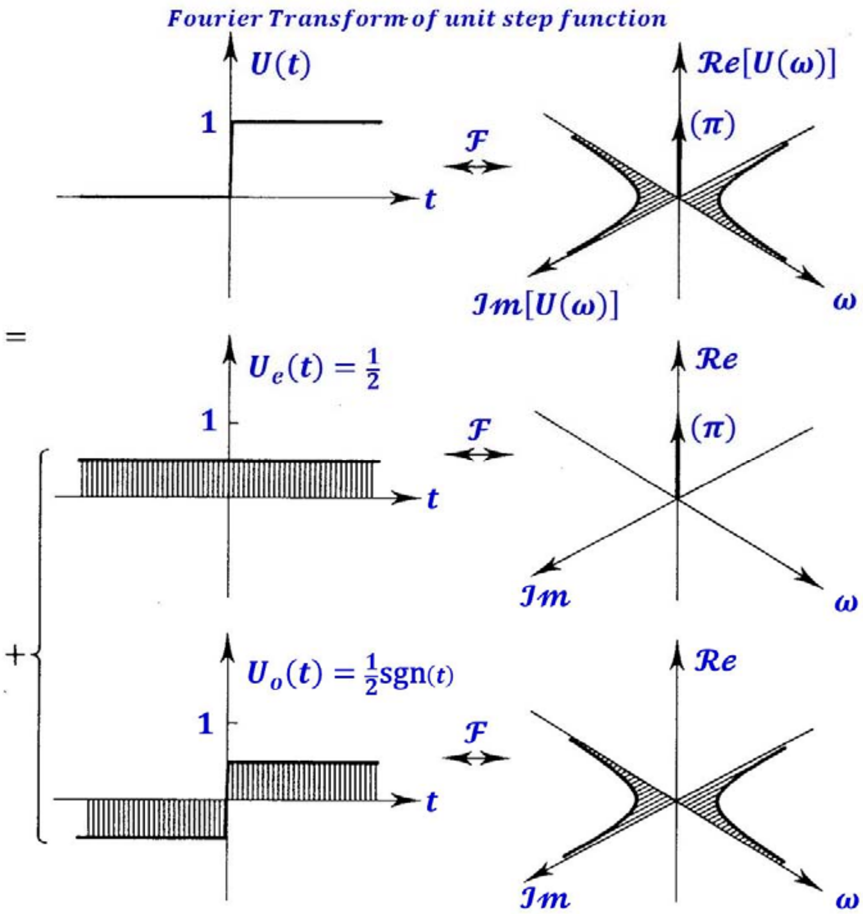
EXAMPLE: Find the transform of the unit step $U(t)$ from the transforms of its even and odd components.

NOTE: In general, **any real function $f(t)$** may be expressed as:

$$f(t) = \underbrace{f_e(t)}_{even} + \underbrace{f_o(t)}_{odd} \quad where \quad \begin{cases} f_e(t) = \frac{1}{2}[f(t) + f(-t)] \\ f_o(t) = \frac{1}{2}[f(t) - f(-t)] \end{cases}$$

Therefore, in our example:

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) \quad \leftrightarrow \quad U(\omega) = \frac{1}{i\omega} + \pi\delta(\omega)$$



TIME SCALING:

Let: $\boldsymbol{f(t) \leftrightarrow F(\omega)}$ then $\boldsymbol{f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)}$

Thus, expansion of the time scale (or time duration) leads to compression of the frequency scale (or bandwidth) and vice versa. This is accompanied by an inverse scaling of the amplitude.

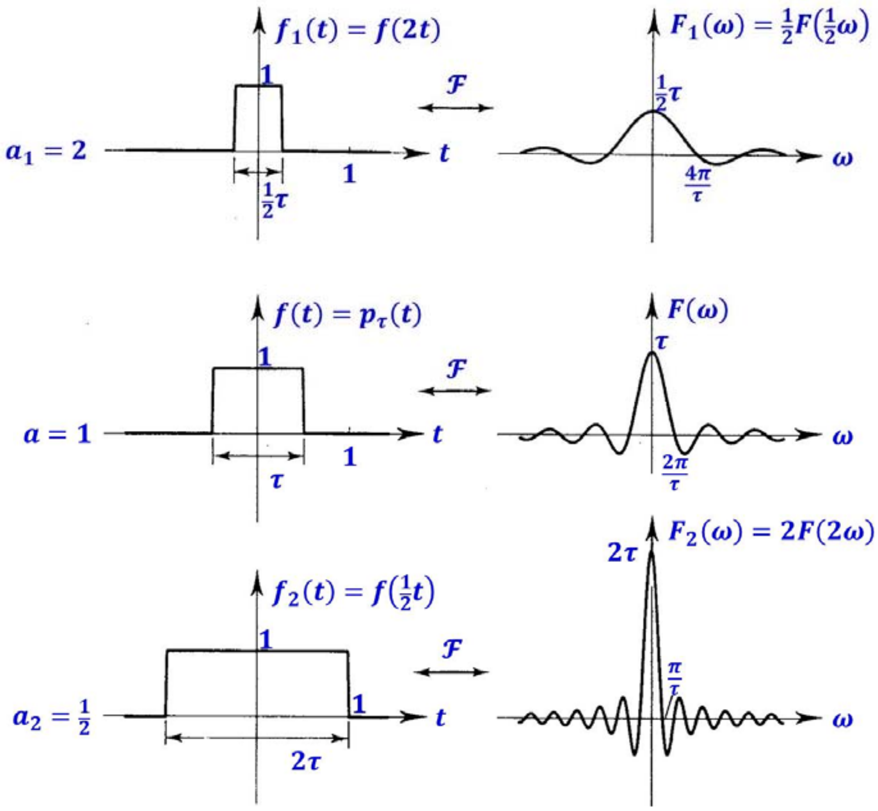
Proof: $\mathcal{F}\{f(at)\} = \int_{-\infty}^{+\infty} f(at) e^{-i\omega t}\, dt$

The change of variable $x = at$, which implies $t = (x/a)$ & $dt = (dx/a)$, yields:

$\mathcal{F}\{f(at)\} = \frac{1}{a}\int_{-\infty}^{+\infty} f(x) e^{-i\left(\frac{\omega}{a}\right)x}\, dx$. This is valid when $a > 0$. For $a < 0$ the integration limits are inverted: $\mathcal{F}\{f(at)\} = \frac{1}{a}\int_{+\infty}^{-\infty} f(x) e^{-i\left(\frac{\omega}{a}\right)x}\, dx = -\frac{1}{a}\int_{-\infty}^{+\infty} f(x) e^{-i\left(\frac{\omega}{a}\right)x}\, dx$. These are combined into the single pair: $f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$.

EXAMPLE:

$$f(t) = p_\tau(t) = \begin{cases} 1 & |t| < \frac{1}{2}\tau \\ 0 & |t| > \frac{1}{2}\tau \end{cases} \quad \leftrightarrow \quad F(\omega) = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$



FREQUENCY SCALING:

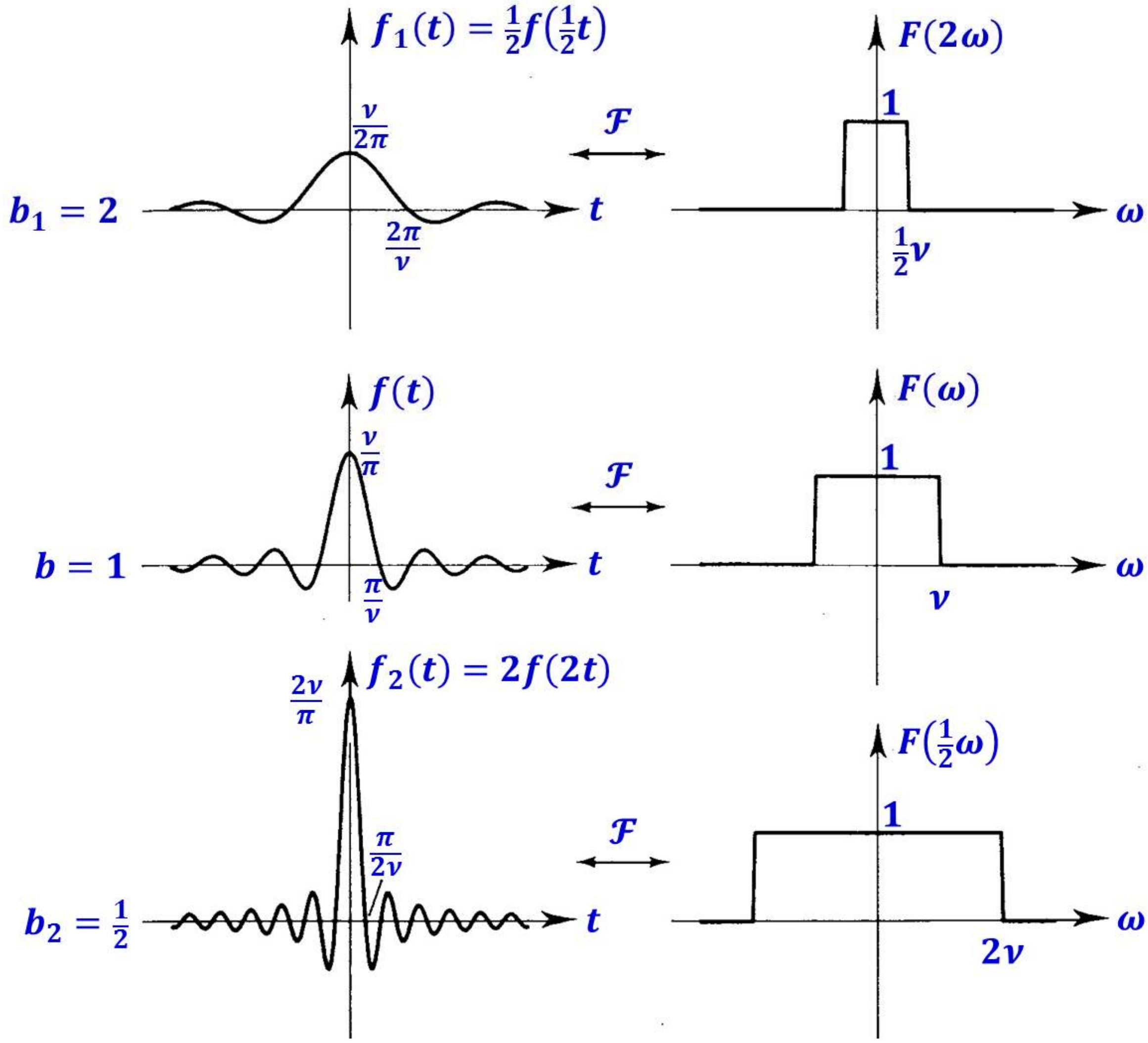
Scaling the frequency variable by a factor b has a similar effect to the scaling. Making the substitution $b = (1/a)$ in expression $f(at) \quad \leftrightarrow \quad \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$ yields the expression:

$$\frac{1}{|b|} f\left(\frac{t}{b}\right) \quad \leftrightarrow \quad F(b\omega)$$

which is completely symmetrical to the expression for **'time scaling'** and, therefore, is an **expression of the duality of the Fourier Transform**.

EXAMPLE:

$$f(t) = \left(\frac{\nu}{\pi}\right) \frac{\sin(\nu t)}{\nu t} \quad \leftrightarrow \quad F(\omega) = p_{2\nu}(\omega)$$



SYMMETRY (DUALITY) OF TRANSFORM:

If $F(\omega)$ is the Fourier Integral of $f(t)$, then:

$$\boldsymbol{F(t) \quad \leftrightarrow \quad 2\pi f(-\omega)}$$

Proof:

The above follows from:

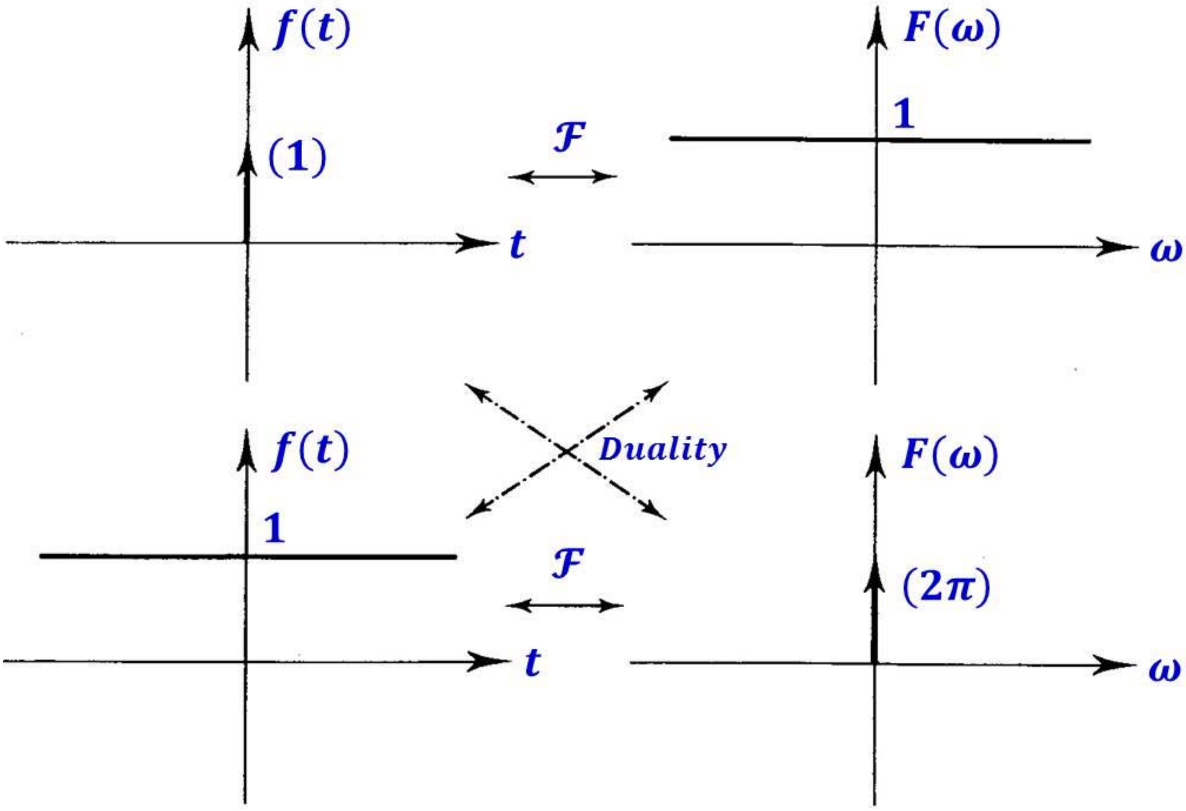
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{+i\omega t} \, d\omega$$

if we write it as follows:

$$2\pi f(-t) = \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega t} \, d\omega$$

and interchange t and ω.

EXAMPLE: From $\delta(t) \leftrightarrow 1$ and the symmetry $\delta(-\omega) = \delta(\omega)$ deduce that $1 \leftrightarrow 2\pi\delta(\omega)$.

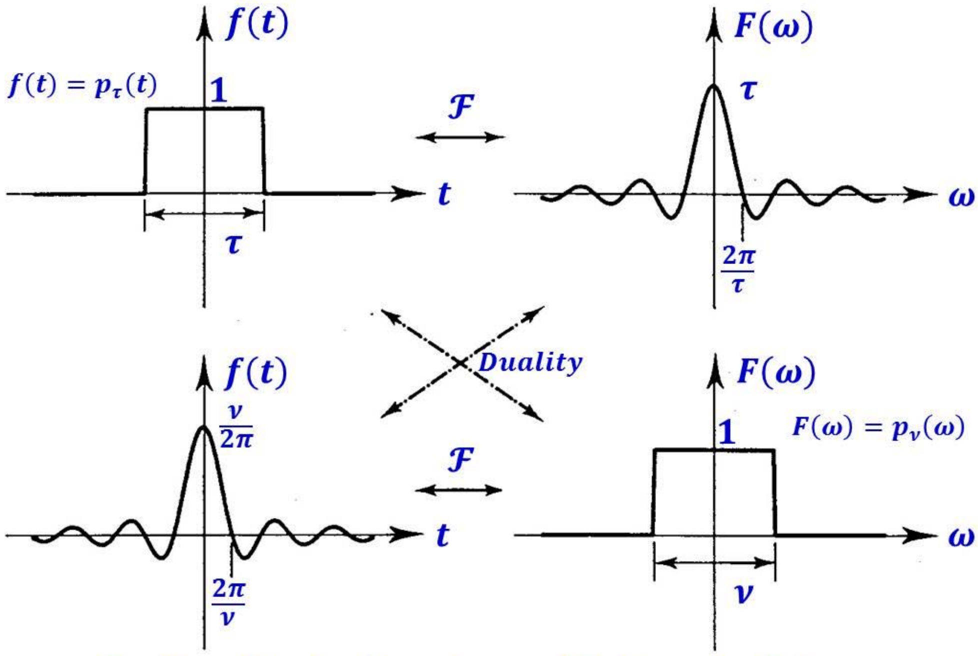


Duality of Fourier Transforms of Impulses

SYMMETRY (DUALITY) OF TRANSFORM: (continued)

EXAMPLE: From knowledge of $p_\tau(t) = \tau\left\{\sin\left(\frac{\omega\tau}{2}\right)/\left(\frac{\omega\tau}{2}\right)\right\}$ find the inverse transform of a rectangular frequency domain pulse:

$$F(\omega) = p_\nu(\omega) = \begin{cases} 1 & |\omega| < \frac{1}{2}\nu \\ 0 & elsewhere \end{cases}$$



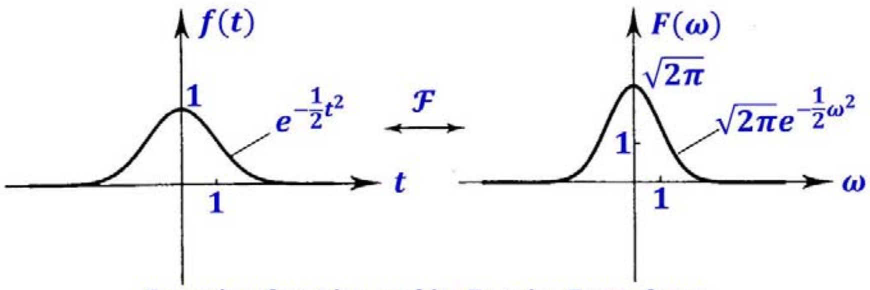
Duality of Fourier Transforms of Rectangular Pulses

EXAMPLE: Gaussian Function (represents the ultimate in symmetry and duality):

$$\boxed{e^{-at^2} \quad \leftrightarrow \quad \sqrt{\frac{\pi}{a}}\, e^{-(\omega^2/4a)}}$$

For $a = \left(\frac{1}{2}\right)$:

$$e^{-\frac{1}{2}t^2} \quad \leftrightarrow \quad \sqrt{2\pi}\, e^{-(\omega^2/2)}$$



Gaussian function and its Fourier Transform

TIME SHIFTING:

Let $\boldsymbol{f(t) \leftrightarrow F(\omega) = A(\omega)e^{i\phi(\omega)}}$ then:

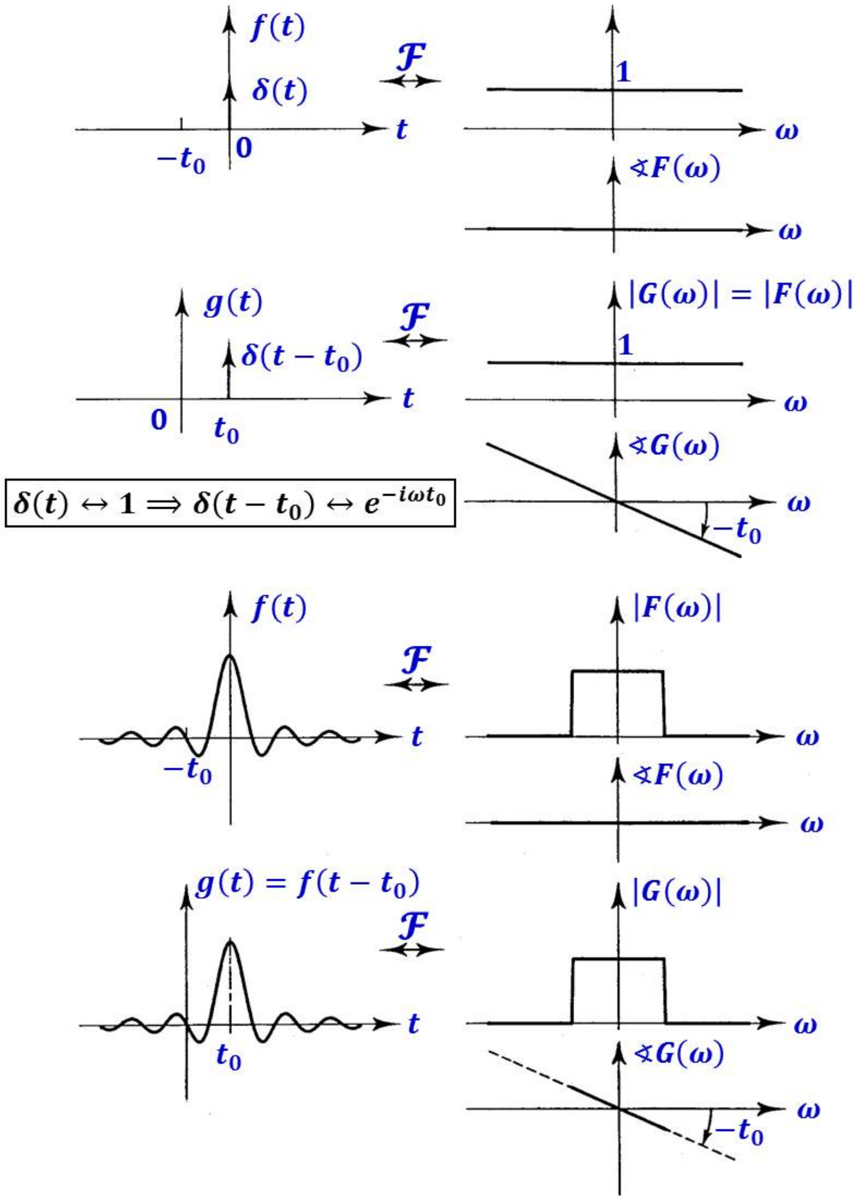
$$\boldsymbol{f(t-t_0) \leftrightarrow F(\omega)e^{-it_0\omega} = A(\omega)e^{i[\phi(\omega)-t_0\omega]}}$$

i.e., if the function $\boldsymbol{f(t)}$ is shifted by a constant, $\boldsymbol{t_0}$**,** then its Fourier spectrum remains the same, but a linear term $\boldsymbol{-t_0\omega}$ is added to its phase angle.

Proof:

$$\int_{-\infty}^{+\infty} f(t-t_0)e^{-i\omega t}\,dt = \int_{-\infty}^{+\infty} f(x)e^{-i\omega(t_0+x)}\,dx = F(\omega)e^{-it_0\omega}$$

EXAMPLE:



FREQUENCY SHIFTING:

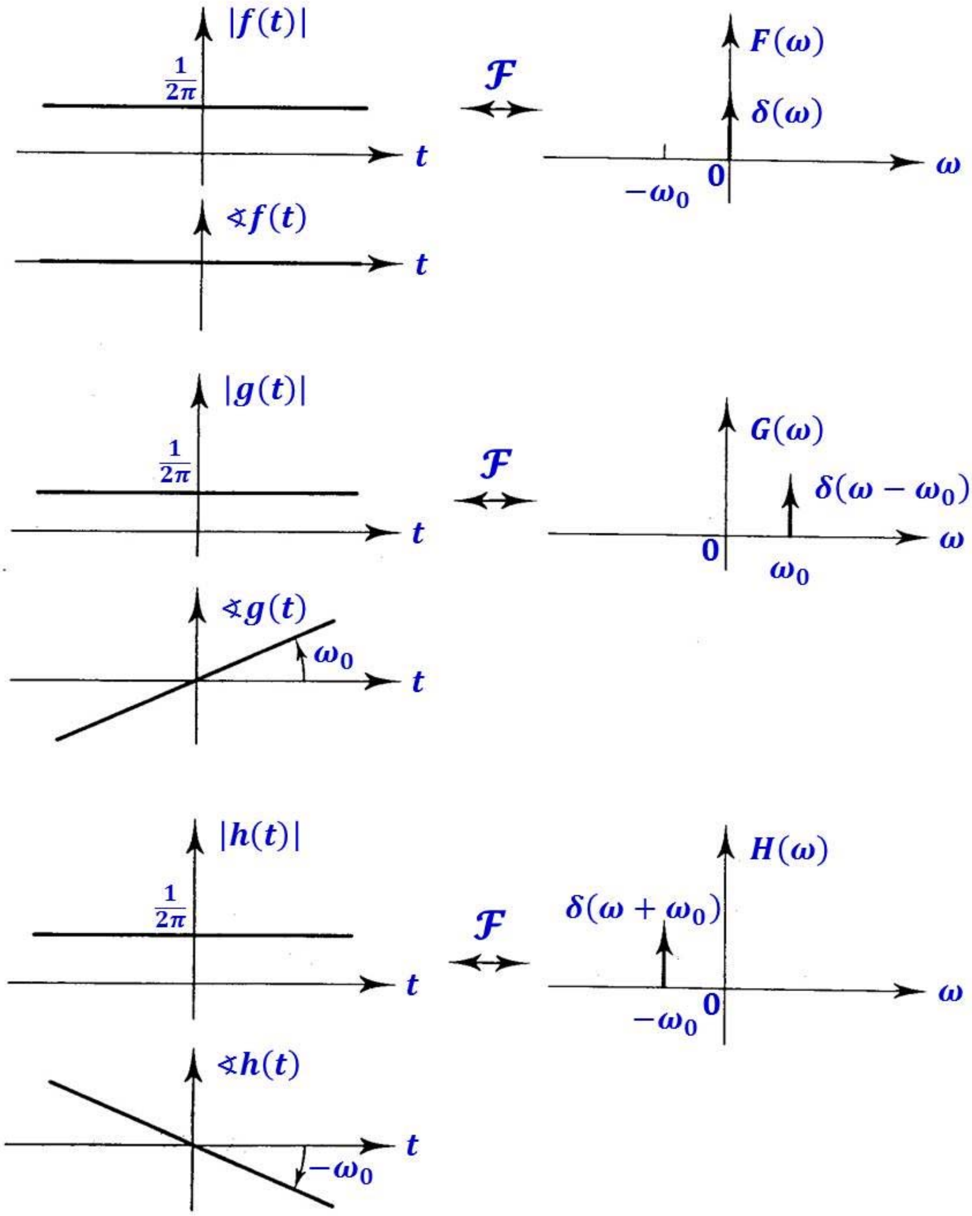
$$\boldsymbol{e^{i\omega_0 t} f(t) \quad \leftrightarrow \quad F(\omega - \omega_0)}$$

Proof:

$$\int_{-\infty}^{+\infty} f(t) e^{i\omega_0 t} e^{-i\omega t}\, dt = \int_{-\infty}^{+\infty} f(t) e^{-i(\omega-\omega_0)t}\, dt = F(\omega - \omega_0)$$

EXAMPLE:

$$\frac{1}{2\pi} \leftrightarrow \delta(\omega) \quad \Longrightarrow \quad \frac{1}{2\pi} e^{i\omega_0 t} \leftrightarrow \delta(\omega - \omega_0)$$



TIME CONVOLUTION:

The Fourier Transform $\boldsymbol{F(\omega)}$ of the convolution $\boldsymbol{f(t)}$ of two functions $\boldsymbol{f_1(t)}$ & $\boldsymbol{f_2(t)}$ equals the product of the Fourier Transform $\boldsymbol{F_1(\omega)}$ & $\boldsymbol{F_2(\omega)}$ of these two functions. Thus:

If $$\boldsymbol{f_1(t) \leftrightarrow F_1(\omega) \quad \& \quad f_2(t) \leftrightarrow F_2(\omega)}$$

Then $$\boldsymbol{f_1(t) * f_2(t) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\, d\tau \quad \leftrightarrow \quad F_1(\omega)F_2(\omega)}$$

Proof:

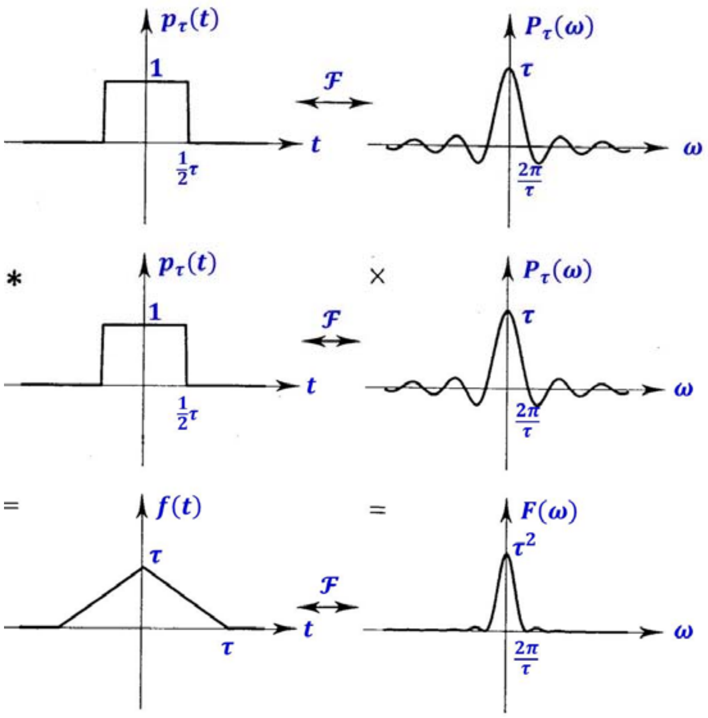
Clearly, $F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t}\left[\int_{-\infty}^{+\infty} f_1(\tau) f_2(t-\tau)\, d\tau\right] dt$

Changing the order of integration, we obtain:

$$F(\omega) = \int_{-\infty}^{+\infty} f_1(\tau) \underbrace{\left[\int_{-\infty}^{+\infty} e^{-i\omega t} f_2(t-\tau)\, dt\right]}_{\substack{F_2(\omega)e^{-i\omega\tau} \\ \textit{Time-shifting Theorem}}} d\tau$$

Therefore: $$F(\omega) = \int_{-\infty}^{+\infty} f_1(\tau) e^{-i\omega\tau} F_2(\omega)\, d\tau = F_1(\omega)F_2(\omega)$$

EXAMPLE:



COMMENT: In the above proof it was assumed that the order of integration can be changed. This is true if the functions $\boldsymbol{f_1(t)}$ & $\boldsymbol{f_2(t)}$ are **square-integrable** in the sense $\int_{-\infty}^{+\infty} |\boldsymbol{f_i(t)}|^2\, \boldsymbol{dt} < \infty$ $(\boldsymbol{i = 1, 2})$, *i.e.*, $\boldsymbol{f_1(t)}$ & $\boldsymbol{f_2(t)}$ have finite energy.

FREQUENCY CONVOLUTION:

From **the time convolution theorem** and **the symmetry (duality) theorem** it follows that the Fourier Transform $\boldsymbol{F(\omega)}$ of the product $\boldsymbol{f_1(t)f_2(t)}$ of two functions equals the convolution $\boldsymbol{F_1(\omega) * F_2(\omega)}$ of their respective derivatives divided by (2π) :

$$\boldsymbol{f_1(t)f_2(t) \quad \leftrightarrow \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(\xi)F_2(\omega-\xi)\, d\xi}$$

One could also give a direct proof as in the time-convolution theorem.

TIME DIFFERENTIATION:

$$\text{Let} \quad \boldsymbol{f(t) \leftrightarrow F(\omega)} \quad \text{then} \quad \boldsymbol{\frac{d^n f(t)}{dt^n} \leftrightarrow (i\omega)^n F(\omega)}$$

For $n = 1 \;\Rightarrow\; \frac{df(t)}{dt} \leftrightarrow (i\omega)F(\omega)$ *i.e.*, **a time differentiation therefore causes a frequency domain rotation $\left(+\frac{\pi}{2}\right)$ of $F(\omega)$, and a linear scaling by ω.**

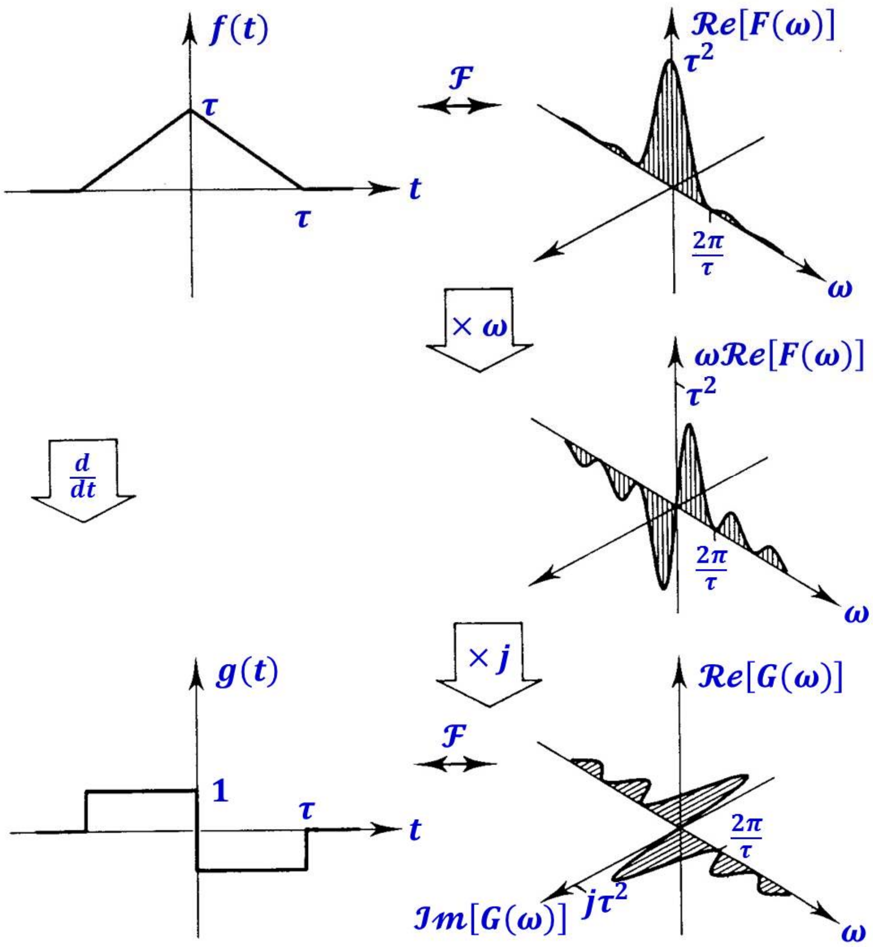
Proof:

Taking the n^{th} derivative of both sides of $f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\, d\omega$ leads to:

$$\frac{d^n f(t)}{dt^n} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} [(i\omega)^n F(\omega)]e^{i\omega t}\, d\omega \quad \Rightarrow \quad \frac{d^n f(t)}{dt^n} \leftrightarrow (i\omega)^n F(\omega)$$

FREQUENCY DIFFERENTIATION:

$$\text{Let} \quad \boldsymbol{f(t) \leftrightarrow F(\omega)} \quad \text{then} \quad \boldsymbol{(-it)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}}$$



INTEGRATION:

Let $\boldsymbol{f(t) \leftrightarrow F(\omega)}$ then $\displaystyle\boldsymbol{\int_{-\infty}^{t} f(\tau)\, d\tau \leftrightarrow \frac{1}{i\omega} F(\omega) + \pi\delta(\omega)F(0)}$

Proof:

We interpret the integral of $f(t)$ as a convolution with the unit step function $U(t)$, *i.e.*

$$g(t) = \int_{-\infty}^{t} f(\tau)\, d\tau = U(t) * f(t)$$

Recall that:

$$U(t) = \frac{1}{i\omega} + \pi\delta(\omega)$$

Let:

$$g(t) \leftrightarrow G(\omega)$$

Then, invoking **the convolution in time theorem**, we get:

$$G(\omega) = U(\omega)F(\omega) = \frac{1}{i\omega} F(\omega) + \pi\delta(\omega)F(0)$$

The term $[\boldsymbol{F(\omega)/(i\omega)}]$ represents the inverse of the differentiation property. **If $F(0) = 0$, then the properties (of differentiation and integration) are fully recoverable, in the sense that the function $G(\omega)$ can be recovered from $F(\omega)$ through division by $(i\omega)$**.

In contrast**, if the function $f(t)$ contains a non-zero $d.c.$ component, represented by a non-zero value of $F(0)$, the transform of its integral contains an additional impulse of strength $\pi F(0)$ at the origin**.

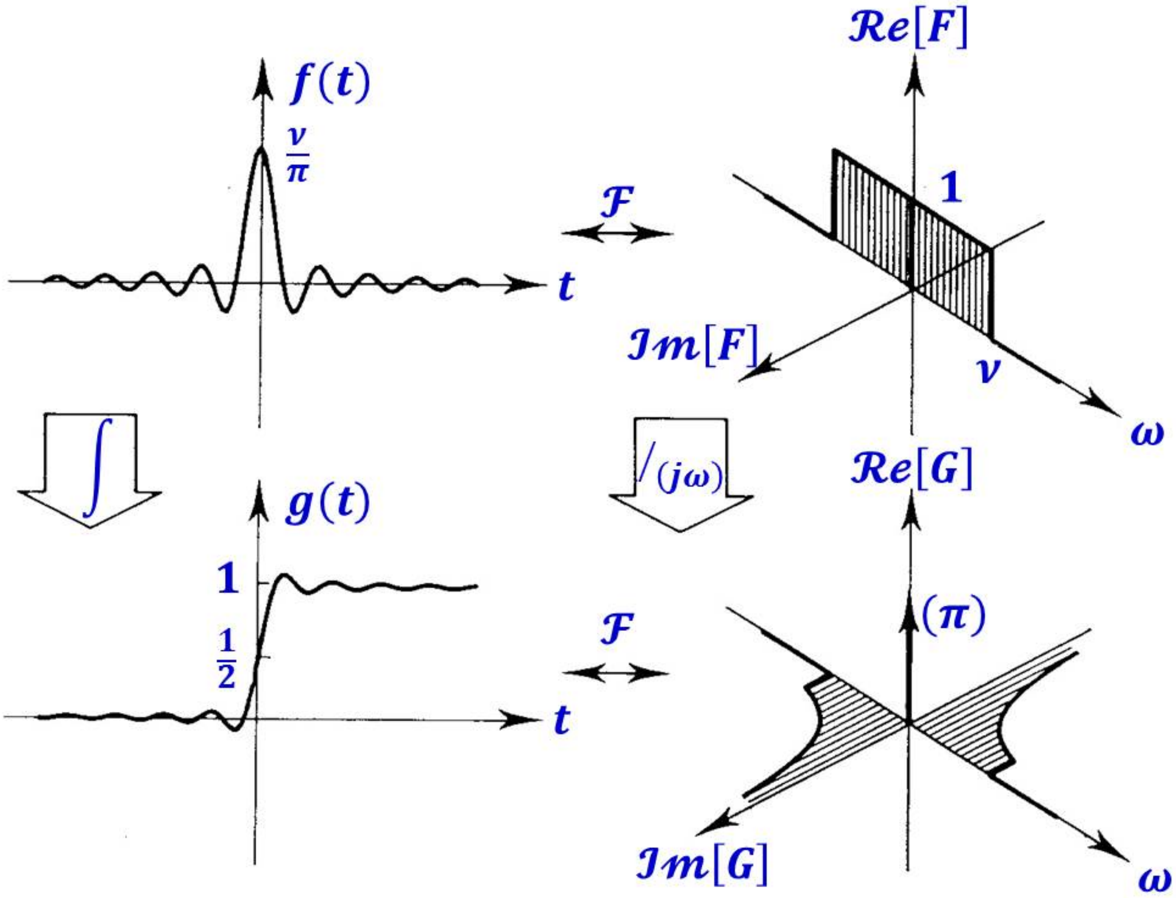
EXAMPLE:

$$f(t) = \left(\frac{\nu}{\pi}\right)\frac{\sin(\nu t)}{\nu t} \quad \leftrightarrow \quad F(\omega) = p_{2\nu}(\omega)$$

The $\boldsymbol{d.c.}$ value of the function is $\boldsymbol{F(0) = 1}$. Thus,:

$$g(t) = \int_{-\infty}^{t} f(\tau)\,d\tau = \int_{-\infty}^{t} \left(\frac{\nu}{\pi}\right)\frac{\sin(\nu t)}{\nu t}\,d\tau \quad \leftrightarrow \quad G(\omega) = -\frac{i}{\omega}p_{2\nu}(\omega) + \pi\delta(\omega)$$

The frequency representation consists of a hyperbola in the imaginary plane, truncated by the pulse $\boldsymbol{p_{2\nu}(\omega)}$, and **a real impulse at the origin of magnitude $\boldsymbol{\pi}$. This impulse transforms back to the time domain as the constant $\boldsymbol{(1/2)}$, which represents the $\boldsymbol{d.c.}$ value of $\boldsymbol{g(t)}$**. Even symmetry of $\boldsymbol{f(t)}$ makes this value coincide with the value at the origin, $\boldsymbol{g(0) = (1/2)}$, and leads to the asymptotic value $\boldsymbol{g(\infty) = 1}$.



EXAMPLE:

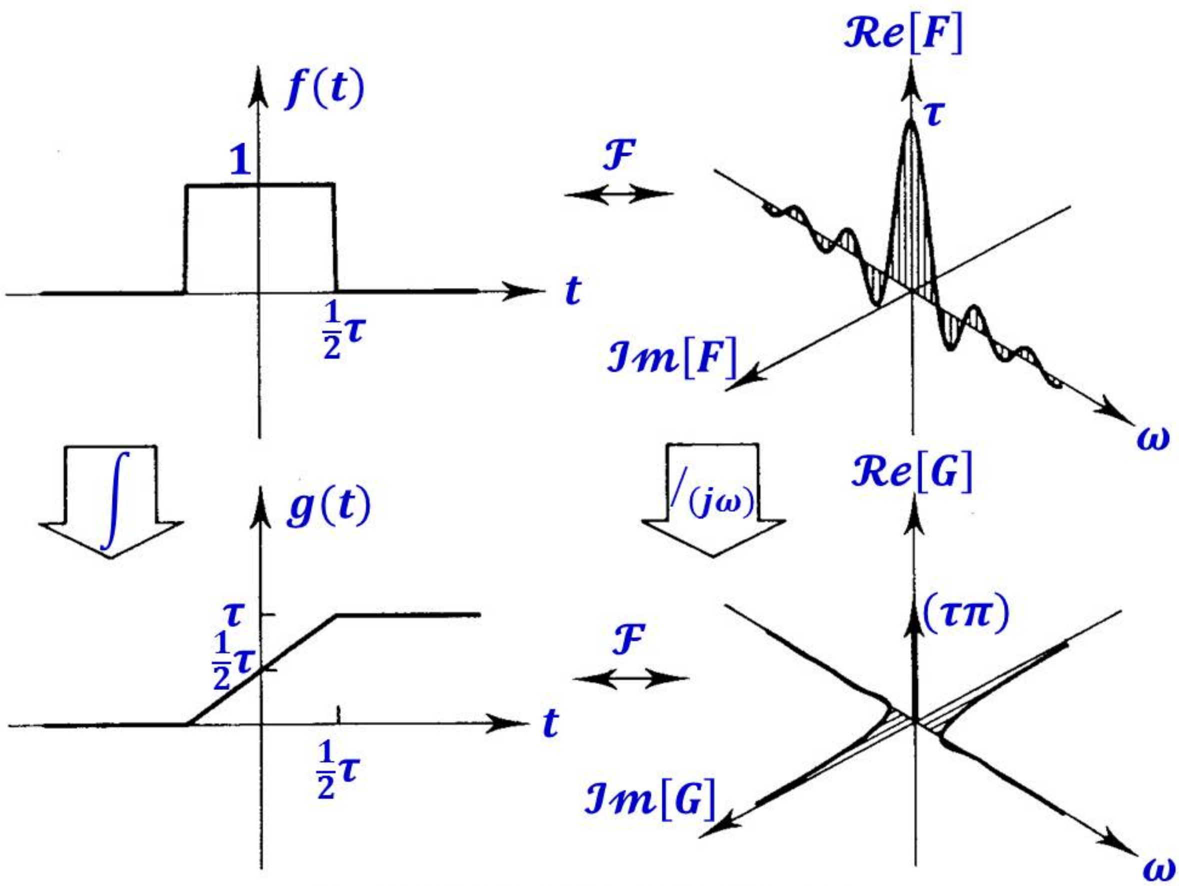
Let:

$$f(t) = p_\tau(t) \quad \leftrightarrow \quad F(\omega) = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

The $\boldsymbol{d.c.}$ value of this function is $\boldsymbol{F(0) = \tau}$. Thus:

$$\boxed{g(t) = \int_{-\infty}^{t} f(\tau)\,d\tau = \int_{-\infty}^{t} p_\tau(\tau)\,d\tau \quad \leftrightarrow \quad G(\omega) = -2i\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega^2} + \pi\tau\delta(\omega)}$$

The integral of $\boldsymbol{f(t)}$ is a truncated ramp, whose value at the origin $g(0) = \frac{1}{2}\tau$ corresponds to **the function's $\boldsymbol{d.c.}$ value and related to the frequency domain impulse of strength $\boldsymbol{\pi\tau}$**.



Fourier Transform of Integral

CONJUGATE FUNCTIONS:

$$\text{Let} \quad \boldsymbol{f(t) \leftrightarrow F(\omega)} \quad \text{then} \quad \boldsymbol{f^*(t) \leftrightarrow F^*(-\omega)}$$

i.e., the Fourier Transform of the conjugate $\boldsymbol{f^*(t) = f_1(t) - if_2(t)}$ of a complex function $\boldsymbol{f(t) = f_1(t) + if_2(t)}$ is given by $\boldsymbol{F^*(-\omega)}$.

Proof:

From

$$F(\omega) = \int_{-\infty}^{+\infty} (f_1 + if_2)e^{-i\omega t}\, dt$$

$$\Longrightarrow \quad F^*(\omega) = \int_{-\infty}^{+\infty} (f_1 - if_2)e^{i\omega t}\, dt$$

$$\Longrightarrow \quad F^*(-\omega) = \int_{-\infty}^{+\infty} (f_1 - if_2)e^{-i\omega t}\, dt$$

$$\Longrightarrow \quad f^*(t) \leftrightarrow F^*(-\omega)$$

PARSEVAL's FORMULA:

If $\boldsymbol{f(t) \leftrightarrow F(\omega) = A(\omega)e^{i\phi(\omega)}}$ then:

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A^2(\omega)\, d\omega$$

Proof:

$$f(t) \leftrightarrow F(\omega) \quad \Longrightarrow \quad f^*(t) \leftrightarrow F^*(-\omega) \quad (\textbf{conjugate functions})$$

Therefore:

$$|f(t)|^2 = f(t)f^*(t) \quad \leftrightarrow \quad \frac{1}{2\pi}F(\omega) * F^*(-\omega) \quad (\textbf{frequency convolution})$$

i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(y)F^*[-(\omega - y)]\, dy \quad = \quad \int_{-\infty}^{+\infty} |f(t)|^2 e^{-i\omega t}\, dt$$

$$\underbrace{\Longrightarrow}_{\omega=0} \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{F(y)F^*(y)}_{A^2(y)}\, dy \quad = \quad \int_{-\infty}^{+\infty} |f(t)|^2\, dt$$

Therefore:

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A^2(\omega)\, d\omega$$

The following is **a more general form of PARSEVAL's formula**:

If $\boldsymbol{f_1(t) \leftrightarrow F_1(\omega)}$ & $\boldsymbol{f_2(t) \leftrightarrow F_2(\omega)}$ then:

$$\int_{-\infty}^{+\infty} f_1(t)f_2(t)\, dt \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(-\omega)F_2(\omega)\, d\omega$$

If $\boldsymbol{f_1(t)}$ & $\boldsymbol{f_2(t)}$ are **real functions** then:

$$\int_{-\infty}^{+\infty} f_1(t)f_2(t)\, dt \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1^*(\omega)F_2(\omega)\, d\omega$$

NOTE: $\boldsymbol{A^2(\omega)}$ = ***energy spectrum*** **of** $\boldsymbol{f(t)}$

$\boldsymbol{E_{12}(\omega) = F_1^*(\omega)F_2(\omega)}$ = ***cross-energy spectrum*** **of** $\boldsymbol{f_1(t)}$ & $\boldsymbol{f_2(t)}$

FOURIER TRANSFORMS INVOLVING IMPULSES:

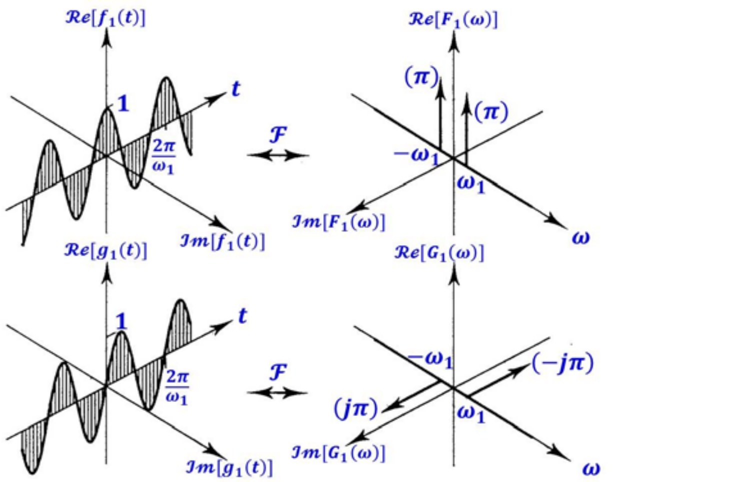
We derive the Fourier Transforms of the time domain functions:

$$f_1(t) = \cos(\omega_1 t) \quad \& \quad g_1(t) = \sin(\omega_1 t)$$

We can easily derive: $\frac{1}{2}e^{i\omega_1 t} \leftrightarrow \pi\delta(\omega - \omega_1) \quad \& \quad \frac{1}{2}e^{-i\omega_1 t} \leftrightarrow \pi\delta(\omega + \omega_1)$

Therefore: $f_1(t) = \cos(\omega_1 t) = \frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2} \quad \leftrightarrow \quad \pi\delta(\omega - \omega_1) + \pi\delta(\omega + \omega_1)$

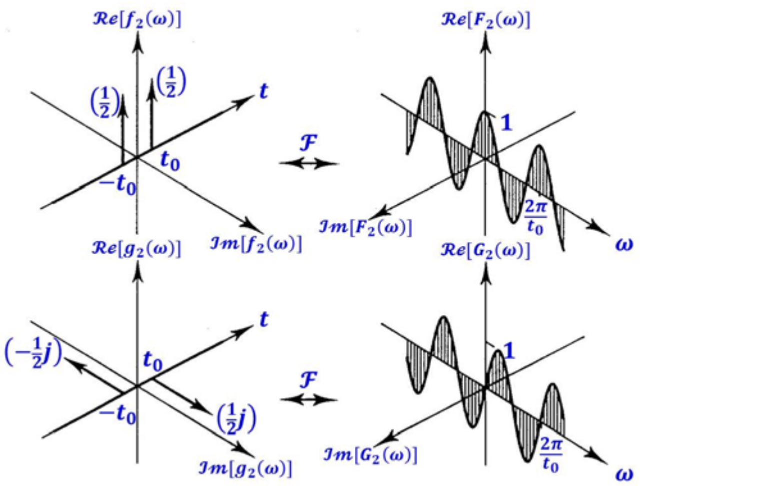
and: $g_1(t) = \sin(\omega_1 t) = \frac{e^{i\omega_1 t} - e^{-i\omega_1 t}}{2i} \quad \leftrightarrow \quad -i\pi\delta(\omega - \omega_1) + i\pi\delta(\omega + \omega_1)$



$$\frac{1}{2}\delta(t - t_0) \leftrightarrow \frac{1}{2}e^{-i\omega t_0} \qquad \& \qquad \frac{1}{2}\delta(t + t_0) \leftrightarrow \frac{1}{2}e^{i\omega t_0}$$

$$f_2(t) = \frac{1}{2}\delta(t - t_0) + \frac{1}{2}\delta(t + t_0) \quad \leftrightarrow \quad F_2(\omega) = \cos(\omega t_0)$$

$$g_2(t) = i\frac{1}{2}\delta(t - t_0) - i\frac{1}{2}\delta(t + t_0) \quad \leftrightarrow \quad G_2(\omega) = \sin(\omega t_0)$$



APPENDIX

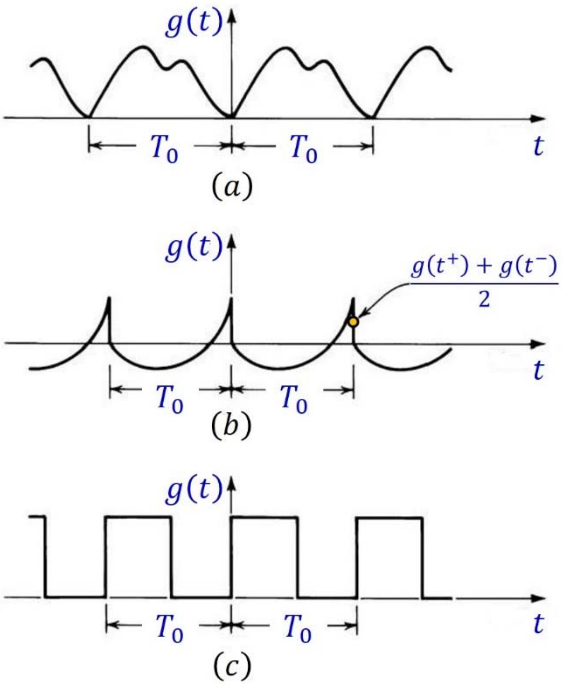
FOURIER SERIES REPRESENTATION OF A PERIODIC FUNCTION

A function $g(t)$ of time t is said to be a ***periodic function*** of t with a period equal to T_0 if it satisfies the following relationship

$$g(t + nT_0) \quad = \quad g(t)$$

where n is an integer with negative or positive values.

The following FIGURE presents several examples of periodic functions.



A periodic function with a **finite number of discontinuities** and a **finite number of maxima or minima within a range of time equal to its period** T_0 can be represented by an infinite trigonometric series as follows

$$g(t) \quad = \quad a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

where a_n and b_n are constants to be determined and $f_0 = (1/T_0)$ is the ***frequency*** in **cycles per second**. The trigonometric series of the above Equation is known as the ***Fourier series***. Setting $\omega_0 = 2\pi f_0$, where ω_0 is the ***circular frequency*** in **radians per second**, the series can be expressed in the alternative form

$$g(t) \quad = \quad a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

To obtain the coefficients a_n, we multiply the left- and the right-hand sides of the above Equation by $\cos(m\omega_0 t)$ and **integrate over a period**. Then noting that

$$\int_{-T_0/2}^{T_0/2} \cos(n\omega_0 t)\cos(m\omega_0 t)\;dt \;=\; \begin{cases} 0 & \text{for } n \neq m \\ T_0/2 & \text{for } n = m \end{cases}$$

$$\int_{-T_0/2}^{T_0/2} \cos(m\omega_0 t)\;dt \;=\; \begin{cases} 0 & \text{for } m \neq 0 \\ T_0 & \text{for } m = 0 \end{cases}$$

$$\int_{-T_0/2}^{T_0/2} \sin(n\omega_0 t)\cos(m\omega_0 t)\;dt \;=\; 0 \quad \text{for all } m \text{ and } n$$

we obtain the following expression for coefficients a:

$$\boxed{\begin{aligned} a_0 &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt \\ a_n &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\cos(n\omega_0 t)\;dt \end{aligned}}$$

Coefficients b_n are obtained in a similar manner by multiplying the left- and the right-hand sides of the Fourier series expansion (*i.e.* the last boxed equation shown above) by $\sin(n\omega_0 t)$ and integrating over a period. Then, because of the relationships

$$\int_{-T_0/2}^{T_0/2} \sin(n\omega_0 t)\sin(m\omega_0 t)\;dt \;=\; \begin{cases} 0 & \text{for } n \neq m \\ T_0/2 & \text{for } n = m \end{cases}$$

$$\int_{-T_0/2}^{T_0/2} \sin(n\omega_0 t)\cos(m\omega_0 t)\;dt \;=\; 0 \quad \text{for all } m \text{ and } n$$

the coefficient b_n is obtained as

$$\boxed{b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\sin(n\omega_0 t)\;dt}$$

It can be shown that at a discontinuity such as the one shown in the FIGURE (above), the ***Fourier series*** expansion converges to a value that is the average of the values of the function $g(t)$ immediately to the left and the right of the discontinuity.

NOTE: The coefficients a_0, a_n and b_n are called ***Fourier coefficients*** of $g(t)$.

COMPLEX EXPONENTIAL FORM OF THE FOURIER SERIES

In developing the frequency-domain analysis procedure, it is convenient first to obtain an exponential form for the ***Fourier series*** (see boxed equations above). Using ***Euler's formula***, the sine and cosine functions in the ***Fourier series*** expansion can be expressed in terms of complex exponentials as follows

$$\sin(n\omega_0 t) = \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}$$
$$\cos(n\omega_0 t) = \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}$$

Substitution of the second of the above expressions in the expression for X (shown in the box above) yields

$$a_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\left(e^{in\omega_0 t} + e^{-in\omega_0 t}\right) dt$$

If we replace n by $-n$ in the above equation, we get

$$a_{-n} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\left(e^{-in\omega_0 t} + e^{in\omega_0 t}\right) dt = a_n$$

In a similar manner, b_n is obtained as follows

$$b_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{i} dt$$
$$b_{-n} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\frac{e^{-in\omega_0 t} - e^{in\omega_0 t}}{i} dt = -b_n$$

Then, the ***Fourier series*** expansion can be expressed as

$$\begin{aligned}
g(t) &= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i} \\
&= a_0 + \sum_{n=1}^{\infty} e^{in\omega_0 t}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) + \sum_{n=1}^{\infty} e^{-in\omega_0 t}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right) \\
&= a_0 + \sum_{n=1}^{\infty} e^{in\omega_0 t}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) + \sum_{n=-\infty}^{-1} e^{in\omega_0 t}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right) \\
&= a_0 + \sum_{n=1}^{\infty} e^{in\omega_0 t}\underbrace{\frac{1}{2}(a_n - b_n)}_{c_n} + \sum_{n=-\infty}^{-1} e^{in\omega_0 t}\underbrace{\frac{1}{2}(a_n - b_n)}_{c_n} \\
&= \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 t}
\end{aligned}$$

Evidently, the coefficient c_n is given by

$$\begin{aligned}
c_n &= \frac{1}{2}(a_n - b_n) \\
&= \frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} g(t)\left\{\left(e^{in\omega_0 t} + e^{-in\omega_0 t}\right) - \left(e^{in\omega_0 t} - e^{-in\omega_0 t}\right)\right\} dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)e^{-in\omega_0 t} dt
\end{aligned}$$

Evidently,

$$c_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\,dt = a_0$$

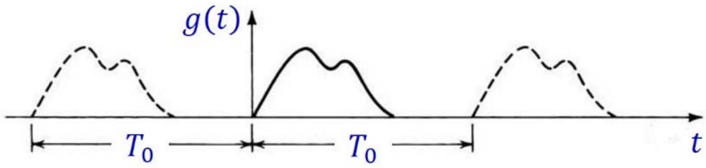
Summarizing, the **complex exponential form** of the ***Fourier series*** is

$$g(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 t}$$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)e^{-in\omega_0 t}\,dt$$

HEURISTIC ARGUMENT TO OBTAIN THE FOURIER INTEGRAL REPRESENTATION OF A NON-PERIODIC FUNCTION FROM FOURIER SERIES

To obtain a Fourier representation of a non-periodic function, we first construct a periodic version of the given function. The first step is to select a value for the period. The selected period should, of course, be larger than the duration of the function. Within each period, the periodic version has a magnitude equal to the specified function for the duration of the latter but is zero otherwise. A periodic version constructed as above is shown in the FIGURE (below).



It is evident that the curves shown by dashed lines represent fictitious replicas of the function that, in fact, do not exist. If T_0 is now increased to a very large value, the fictitious replicas of the function will move to infinity and in the limit we will get a true representation of the given function. We apply this reasoning to derive a Fourier representation of a non-periodic function. [NOTE: If the non-periodic function has infinite duration, we consider the Fourier series representation of finite duration segment of it, $(-T_0/2\,,T_0/2)$, which in the limit, as $T_0 \to +\infty$, leads to the original function.]

First since T_0 is very large, we set

$$\omega_0 = \frac{2\pi}{T_0} = \Delta\Omega$$

and

$$n\omega_0 = \Omega_n$$

With this notation we get

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)e^{-i\Omega_n t}\,dt \;\Rightarrow\; c_n T_0 = \int_{-T_0/2}^{T_0/2} g(t)e^{-i\Omega_n t}\,dt$$

Thus

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 t} \\ &= \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} c_n T_0 e^{i\Omega_n t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} c_n T_0 e^{i\Omega_n t} \Delta\Omega \end{aligned}$$

In the limit as $T_0 \to +\infty$, $\Omega_n = n \cdot \Delta\Omega$ becomes a continuous variable, *i.e.* $\Omega_n \to \Omega$, and $c_n T_0 \to G(\Omega)$, *i.e.*

$$c_n T_0 \quad \to \quad \boxed{G(\Omega) = \int_{-\infty}^{+\infty} g(t) e^{-i\Omega t}\, dt}$$

while

$$g(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} c_n T_0 e^{i\Omega_n t} \Delta\Omega \quad \to \quad \boxed{g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\Omega) e^{i\Omega t}\, d\Omega}$$

Evidently, of the above two boxed equations, the first represents the ***Fourier transform*** of the function $g(t)$, and the second equation is the ***inverse Fourier transform***.

The above heuristic argument is mathematically problematic (see the discussion of section 1-1, pages 1 & 2, in PAPOULIS 1962).