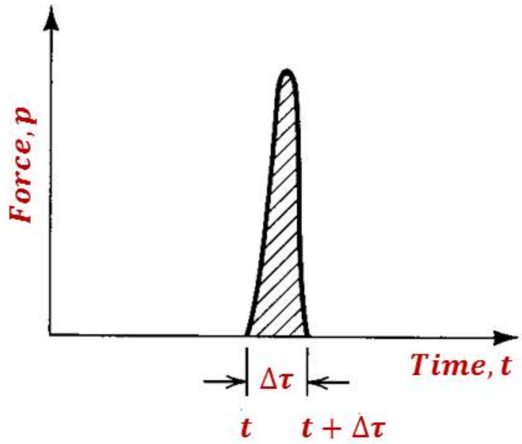
RESPONSE TO GENERAL DYNAMIC LOADING



The time integral of a force is referred to as **impulse**, is determined by $\boldsymbol{I}$ and is obtained from:

$$I = \int_{t}^{t+\Delta t} p(t)\, dt$$

Newton's 2nd Law of motion states that **the action of an (impulsive) force on a mass, results in a change in the velocity of the mass and hence in its linear momentum, the change in linear momentum being equal to the impulse of the (impulsive) force**.

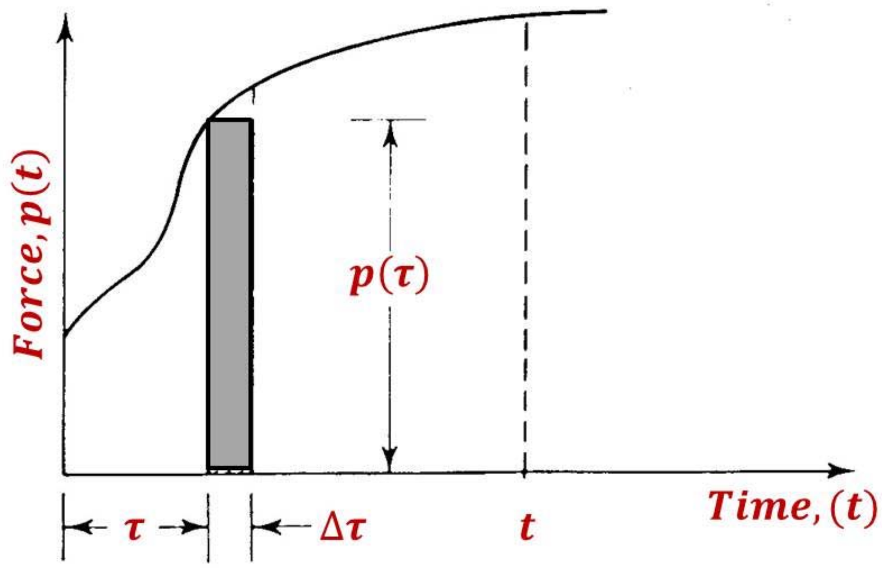
Thus, representing the change in velocity by $\boldsymbol{\Delta\dot{u}}$,

$$m\Delta\dot{u} = I$$

If the mass is **initially at rest**, it will have a velocity $(\boldsymbol{I/m})$ after the action of the impulse.

Suppose that a SDOF system is subjected to an impulse $\boldsymbol{I = p(\tau)\cdot\Delta\tau}$. The action of the impulse will set the system vibrating. **The ensuing free vibration response can be obtained by recognizing that the initial displacement is zero and the initial velocity is** $(\boldsymbol{I/m})$. Thus, the resulting response is:

$$u(t) = \frac{I}{m\omega_d} e^{-\xi\omega t} \sin(\omega_d t) = I\cdot h(t)$$



We approximate the general loading $\boldsymbol{p(t)}$ by a series of pulses of intensity $\boldsymbol{p(t)\cdot\Delta\tau}$.

Hence, a pulse applied at time τ contributes to the response at time t an amount equal to:

$$\Delta u(t) \cong [p(\tau)\Delta\tau]\cdot h(t-\tau)$$

where: $$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega t}\sin(\omega_d t) \quad \left(= \textit{unit impulse response function}\right)$$

The above expression is approximate when $\Delta\tau$ is finite but becomes exact as $\Delta\tau \to 0$.

Thus, the contribution of all the pulses $(\boldsymbol{0 \le \tau \le t})$ is given by:

$$u(t) \cong \sum \{p(\tau)\Delta\tau \cdot h(t-\tau)\}$$

At the limit as $\boldsymbol{\Delta\tau \to 0}$, we obtain:

$$\boxed{u(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau}$$

We can obtain the **unit impulse response** $h(t)$ using the **Dirac (delta) function** $\delta(t)$:

Equation of Motion:

$$m\ddot{u} + c\dot{u} + ku = I\delta(t) \qquad [I = \textit{impulse intensity}]$$

$$\Longrightarrow \quad \ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \left(\frac{I}{m}\right)\delta(t)$$

$$UNITS: \begin{cases} [I] = [F][T] \\ [\delta(t)] = [T]^{-1} \end{cases}$$

Initial conditions: $u(0^-) = 0$ & $\dot{u}(0^-) = 0$ (*i.e.*, system at rest)

Integrate the Equation of Motion formally over $(-\varepsilon, +\varepsilon)$ and take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0}\left\{\int_{-\varepsilon}^{+\varepsilon} \ddot{u}(t)\,dt + 2\xi\omega \int_{-\varepsilon}^{+\varepsilon} \dot{u}(t)\,dt + \omega^2 \int_{-\varepsilon}^{+\varepsilon} u(t)\,dt = \left(\frac{I}{m}\right)\int_{-\varepsilon}^{+\varepsilon} \delta(t)\,dt\right\}$$

$$\lim_{\varepsilon \to 0}\left\{[\dot{u}(+\varepsilon) - \dot{u}(-\varepsilon)] + 2\xi\omega[u(+\varepsilon) - u(-\varepsilon)] + \omega^2 \underbrace{\int_{-\varepsilon}^{+\varepsilon} u(t)\,dt}_{0} = \left(\frac{I}{m}\right)\right\}$$

$$\left[\dot{u}(0^+) - \underbrace{\dot{u}(0^-)}_{0}\right] + 2\xi\omega \underbrace{[u(0^+) - u(0^-)]}_{\substack{0 \\ \because\, u(t) = continuous}} + 0 = \left(\frac{I}{m}\right)$$

Therefore:

$$\boxed{\dot{u}(0^+) = \left(\frac{I}{m}\right)}$$

Therefore, we can make the following statement:

$$\left.\begin{array}{c} m\ddot{u} + c\dot{u} + ku = I\delta(t) \\ u(0^-) = 0 \;\; \& \;\; \dot{u}(0^-) = 0 \end{array}\right\} \;\; equivalent \;\; \left\{\begin{array}{c} m\ddot{u} + c\dot{u} + ku = 0 \;\; (t > 0) \\ \dot{u}(0) = \left(\frac{I}{m}\right) \;\; \& \;\; u(0) = 0 \end{array}\right.$$

Therefore, the effect of the impulsive force $p(t) = I\delta(t)$ is to import to the SDOF system an initial velocity equal to $\left({}^{I}/{}_{m}\right)$.

The response of the SDOF system governed by:

Equation of Motion: $m\ddot{u} + c\dot{u} + ku = 0 \;\; (t > 0)$

Initial Conditions: $\dot{u}(0) = \left(\frac{I}{m}\right) \;\; \& \;\; u(0) = 0$

is given by:

$$\begin{aligned} u(t) &= e^{-\xi\omega t}\left[u(0)\cos(\omega_d t) + \frac{\dot{u}(0) + \xi\omega u(0)}{\omega_d}\sin(\omega_d t)\right] \\ &= \frac{\left({}^{I}/{}_{m}\right)}{\omega_d} e^{-\xi\omega t}\sin(\omega_d t) \end{aligned}$$

For $I = 1$, the above response is denoted by $h(t)$ and is referred to as the unit impulse response function.

Therefore:

$$1 \cdot \delta(t) \;\; \rightarrow \;\; \boxed{\begin{array}{c} SDOF \\ (\xi = 0) \end{array}} \;\; \rightarrow \;\; h(t) = \frac{1}{m\omega}\sin(\omega t)$$

$$1 \cdot \delta(t) \;\; \rightarrow \;\; \boxed{\begin{array}{c} SDOF \\ (0 < \xi < 1) \end{array}} \;\; \rightarrow \;\; h(t) = \frac{1}{m\omega_d} e^{-\xi\omega t}\sin(\omega_d t)$$

Assuming that we know the unit impulse response of the **system/structure**, the response integral $\boldsymbol{u(t) = \int_0^t p(\tau)h(t-\tau)\, d\tau \stackrel{\text{def}}{=} p(t) * h(t)}$ may be derived also using a '***Linear Systems Theory***' approach.

Sifting property **of the Dirac (delta) function** $\boldsymbol{\delta(t)}$**:**

$$\int_{-\infty}^{+\infty} \delta(t-t_0) f(t)\, dt = f(t_0)$$

NOTE: The verb '***to sift***' means **to put through a sieve**.

EXCITATION		***RESPONSE***
$\delta(t)$	$\rightarrow \boxed{SDOF} \rightarrow$	$h(t) = \dfrac{1}{m\omega_d} e^{-\xi\omega t} \sin(\omega_d t)$
$\delta(t-\tau)$	$\rightarrow \boxed{SDOF} \rightarrow$	$\underbrace{h(t-\tau)}_{\text{time-invariant system}}$
$\delta(t-\tau)p(\tau)d\tau$	$\rightarrow \boxed{SDOF} \rightarrow$	$\underbrace{h(t-\tau)p(\tau)d\tau}_{\text{linear system}}$
$\underbrace{\int_0^{+\infty} \delta(t-\tau)p(\tau)d\tau}_{\text{sifting property of } \delta(t)}$	$\rightarrow \boxed{SDOF} \rightarrow$	$\underbrace{\int_0^t h(t-\tau)p(\tau)d\tau}_{\text{superposition}}$
$\boldsymbol{p(t)}$	$\rightarrow \boxed{SDOF} \rightarrow$	$\boldsymbol{u(t) = \int_0^t h(t-\tau)p(\tau)d\tau}$

Therefore, **the response of the SDOF system** to an **arbitrary loading** $\boldsymbol{p(t)}$, **starting from rest**, is given by:

$$u(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau \qquad \begin{pmatrix} \textbf{convolution} \\ or \\ \textbf{Duhamel's} \\ integral \end{pmatrix}$$

The above integral is known as the '***convolution integral***' or '***Duhamel's integral***'.

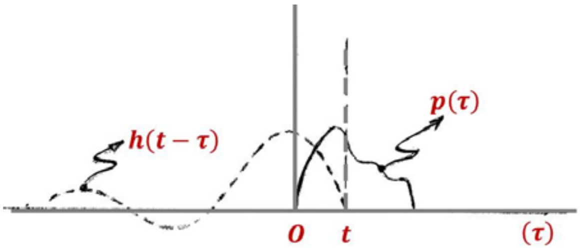
If the SDOF system **starts from a state other than the state of rest**, then the response is given by:

$$u(t) \quad = \quad e^{-\xi\omega t}\left[u(0)\cos(\omega_d t) + \frac{\dot{u}(0) + \xi\omega u(0)}{\omega_d}\sin(\omega_d t)\right] + \int_0^t p(\tau)h(t-\tau)\,d\tau$$

Note the following properties of the convolution integral:

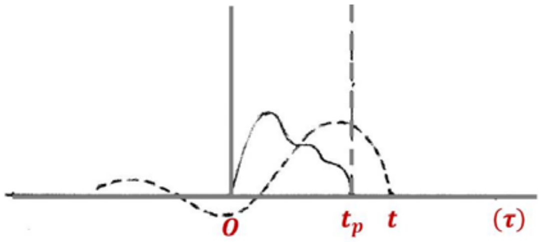
$$u(t) = p(t) * h(t) \stackrel{\text{def}}{=} \int_0^t p(\tau)h(t-\tau)\,d\tau = \int_0^t p(t-\xi)h(\xi)\,d\xi$$

Graphical representation *of* $\int_0^t p(\tau)h(t-\tau)\,d\tau$:



Note that when t is greater than the pulse time, say t_p, then:

$$u(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau = \int_0^{t_p} p(\tau)h(t-\tau)\,d\tau + \underbrace{\int_{t_p}^t p(\tau)h(t-\tau)\,d\tau}_{0} = \int_0^{t_p} p(\tau)h(t-\tau)\,d\tau$$



Considering that for $t > t_p$ the load application ceases, **the oscillator will perform free vibrations with initial conditions** $u(t_p)$ & $\dot{u}(t_p)$. The integral $\int_0^{t_p} p(\tau)h(t-\tau)\,d\tau$ expresses/evaluates these free oscillations.

NOTE:

Let an **Initial Value Problem (IVP)** be specified by the following n^{th} order **linear Ordinary Differential Equation (ODE)** with constant coefficients:

$$a_n \frac{d^n u}{dt^n} + a_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \cdots + a_0 u = r(t)$$

Let the **associated linear operator**:

$$L[\quad] = a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_0$$

Then, the **Green's function** $G(t)$ of the above linear differential operator $L[\quad]$ with constant coefficients is the function that satisfies:

(i) The homogeneous ODE:

$$L[G] = 0$$

(ii) The initial conditions:

$$G(0) = \frac{dG(0)}{dt} = \frac{d^2G(0)}{dt^2} = \cdots = \frac{d^{n-2}G(0)}{dt^{n-2}} = 0 \quad \& \quad \frac{d^{n-1}G(0)}{dt^{n-1}} = \frac{1}{a_n}$$

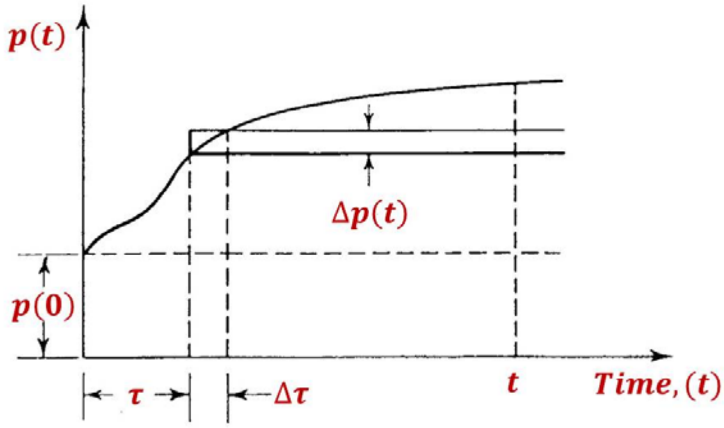
It can be demonstrated that:

$$u(t) = \int_0^t r(\tau) G(t-\tau)\, d\tau$$

is a solution of the above inhomogeneous ODE.

It is now evident that the unit impulse response $h(t)$ is the Green's function of the equation of motion.

Derivation Of Duhamel's (Convolution) Integral
Using the Response to a Force Described by the Unit Step Function



Let $g(t)$ be **the displacement response of a SDOF system**, starting **from rest**, **to a force described by the unit step function** $\mathbb{H}(t)$, i.e., $p(t) = p_0 \cdot \mathbb{H}(t) = 1 \cdot \mathbb{H}(t)$. Specifically,

$$g(t) = \frac{1}{k}\left\{1 - e^{-\xi\omega t}\left[\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right]\right\} \quad , \quad t \geq 0$$

NOTE: The above result may be obtained by considering the governing equation of motion $m\ddot{u} + c\dot{u} + ku = p(t)$, where $p(t) = p_0 \cdot \mathbb{H}(t) = 1 \cdot \mathbb{H}(t)$, subject to initial conditions: $u(t=0) = 0$ & $\dot{u}(t=0) = 0$. The general solution of the above equation is written as follows: $u(t) = u_H(t) + u_P(t) = e^{-\xi\omega t}[A\cos(\omega_d t) + B\sin(\omega_d t)] + (1/k)$. After imposing the given initial conditions, we obtain the above (boxed) result.

Contribution to the response of a step function of amplitude $\Delta p(t,\tau)$ applied at $t = \tau$:

$$\Delta u(t,\tau) \cong \Delta p(\tau) g(t-\tau) = \frac{\Delta p(\tau)}{\Delta\tau} g(t-\tau)\Delta\tau$$

Therefore:

$$u(t) \cong p(0)g(t) + \sum\left\{\frac{\Delta p(\tau)}{\Delta\tau} g(t-\tau)\Delta\tau\right\}$$

As $\Delta\tau \to 0$, we obtain:

$$u(t) = p(0)g(t) + \int_0^t \frac{dp(\tau)}{d\tau} g(t-\tau)\, d\tau$$

Duhamel's integral

Integrating by parts:

$$\begin{aligned} u(t) &= p(0)g(t) + g(t-\tau)p(\tau)|_0^t - \int_0^t p(\tau)\frac{dg(t-\tau)}{d\tau}\,d\tau \\ &= g(0)p(t) + \int_0^t p(\tau)\frac{dg(t-\tau)}{dt}\,d\tau \end{aligned}$$

Applying **Leibnitz rule** for **differentiation under the integral sign**, we obtain:

$$\boxed{u(t) = \frac{d}{dt}\int_0^t g(t-\tau)p(\tau)\,d\tau}$$

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NOTE: LEIBNITZ RULE

Let: $$I(\varepsilon) = \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} f(x,\varepsilon)\,dx$$

Then: $$\frac{dI}{d\varepsilon} = f(x_2,\varepsilon)\frac{dx_2}{d\varepsilon} - f(x_1,\varepsilon)\frac{dx_1}{d\varepsilon} + \int_{x_1(\varepsilon)}^{x_2(\varepsilon)} \frac{\partial f(x,\varepsilon)}{\partial \varepsilon}\,dx$$

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It is straightforward to show that:

$$\boxed{h(t) = \frac{d}{dt}g(t)}$$

Indeed,

$$\frac{dg(t)}{dt} = \frac{d}{dt}\left\{\frac{1}{k} - \frac{e^{-\xi\omega t}}{k}\left[\cos(\omega_D t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_D t)\right]\right\}$$

$$\left\{\begin{array}{c} \dfrac{\xi\omega}{k}e^{-\xi\omega t}\left[\cos(\omega_D t) + \dfrac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_D t)\right] \\ + \\ -\dfrac{1}{k}e^{-\xi\omega t}[-\omega_D\sin(\omega_D t) + \xi\omega\cos(\omega_D t)] \end{array}\right\} = \frac{e^{-\xi\omega t}}{k}\sin(\omega_D t)\left[\frac{\xi^2\omega}{\sqrt{1-\xi^2}} + \omega_D\right]$$

$$\frac{e^{-\xi\omega t}}{k}\sin(\omega_D t)\frac{\omega}{\sqrt{1-\xi^2}} = \frac{e^{-\xi\omega t}}{m\omega_D}\sin(\omega_D t)$$

EXAMPLE:

Let the SDOF system of mass m and stiffness k be subjected to a loading $p(t)$, where:

$$p(t) \quad = \quad \begin{cases} p_0 & t \le t_d \\ 0 & t > t_d \end{cases}$$

Compute the displacement response of the SDOF system using the convolution integral; assume that the system responds to the applied load starting from rest.

Solution:

The unit impulse response for the undamped SDOF system is: $h(t) = \frac{1}{m\omega}\sin(\omega t)$. Therefore:

$$\underbrace{h(t) * p(t)}_{u(t)} \quad = \quad \begin{cases} \int_0^t \frac{1}{m\omega}\sin[\omega(t-\tau)]\cdot p(\tau)\, d\tau & t \le t_d \\ \int_0^{t_d} \frac{1}{m\omega}\sin[\omega(t-\tau)]\cdot p(\tau)\, d\tau & t > t_d \end{cases}$$

$$= \begin{cases} \frac{1}{m\omega^2}\cdot[1-\cos(\omega t)] & t \le t_d \\ \frac{1}{m\omega^2}\cdot\left[\cos\left(\omega(t-t_d)\right)-\cos(\omega t)\right] & t > t_d \end{cases}$$

Details of the evaluation of the convolution integral (two phases):

$$\int_0^t \frac{1}{m\omega}\sin[\omega(t-\tau)]\cdot p(\tau)\, d\tau \quad = \quad \frac{1}{m\omega}\cdot\int_0^t \sin[\omega(t-\tau)]\cdot 1\, d\tau \quad \overset{\xi=t-\tau}{\cong} \quad \frac{1}{m\omega}\cdot\int_t^0 \sin[\omega\xi]\ d(-\xi)$$

$$= \quad \frac{1}{m\omega}\cdot\int_0^t \sin[\omega\xi]\ d\xi \quad \overset{\zeta=\omega\xi}{\cong} \quad \frac{1}{m\omega^2}\cdot\int_0^{\omega t} \sin\zeta\ d\zeta$$

$$= \quad \frac{1}{m\omega^2}\cdot(-\cos\zeta)|_0^{\omega t} \quad = \quad \frac{1}{m\omega^2}\cdot[1-\cos(\omega t)]$$

$$\int_0^{t_d} \frac{1}{m\omega}\sin[\omega(t-\tau)]\cdot p(\tau)\, d\tau \quad = \quad \frac{1}{m\omega}\cdot\int_0^{t_d} \sin[\omega(t-\tau)]\cdot 1\, d\tau \quad \overset{\xi=t-\tau}{\cong} \quad -\frac{1}{m\omega}\cdot\int_t^{t-t_d} \sin[\omega\xi]\ d\xi$$

$$= \quad -\frac{1}{m\omega^2}\cdot\int_{\omega t}^{\omega(t-t_d)} \sin\zeta\ d\zeta \quad = \quad \frac{1}{m\omega^2}\cdot\cos\zeta|_{\omega t}^{\omega(t-t_d)}$$

$$= \quad \frac{1}{m\omega^2}\cdot\left[\cos\left(\omega(t-t_d)\right)-\cos(\omega t)\right]$$

From the result we obtained above for $u(t) = h(t) * p(t)$ by calculating the convolution integral for the time interval $t \leq t_d$, we may calculate the following quantities:

$$\begin{aligned} u(t_d) &= \frac{1}{m\omega^2} \cdot [1 - \cos(\omega t_d)] \\ \dot{u}(t_d) &= \frac{1}{m\omega} \cdot \sin(\omega t_d) \end{aligned}$$

Consider the response the response of the oscillator for $t > t_d$, i.e., **after the load application has ceased**. The oscillator performs ***free vibration oscillations*** with initial conditions $u(t_d)$ & $\dot{u}(t_d)$. Specifically,

$$\begin{aligned} u(t) &= u(t_d)\cos\big(\omega(t - t_d)\big) + \frac{\dot{u}(t_d)}{\omega}\sin\big(\omega(t - t_d)\big) \\ &= \frac{1}{m\omega^2} \cdot \Big[[1 - \cos(\omega t_d)]\cos\big(\omega(t - t_d)\big) + \sin(\omega t_d)\sin\big(\omega(t - t_d)\big)\Big] \\ &= \frac{1}{m\omega^2} \cdot \big[\cos\big(\omega(t - t_d)\big) - \cos(\omega t_d)\cos\big(\omega(t - t_d)\big) + \sin(\omega t_d)\sin\big(\omega(t - t_d)\big)\big] \\ &= \frac{1}{m\omega^2} \cdot \big[\cos\big(\omega(t - t_d)\big) - \cos(\omega t)\big] \end{aligned}$$

The last expression is indeed what we obtained by evaluating the convolution integral for $t > t_d$.