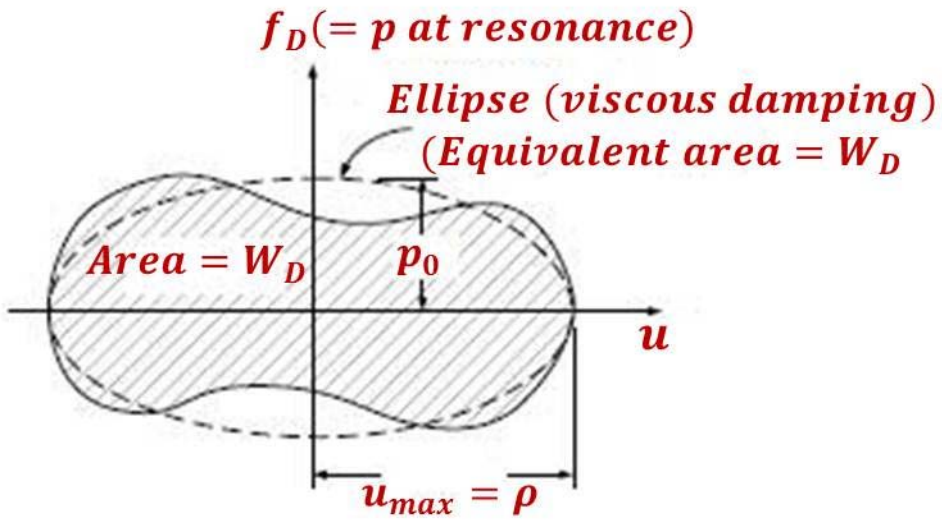
ENERGY DISSIPATED BY DAMPING



$$\boxed{W_D = \oint f_D \, du}$$

Depending on the type of damping present, the dissipative force-displacement relationship when plotted may differ greatly. In all cases, however, **the dissipative force-displacement curve will enclose an area**, referred to as the ‘***hysteretic loop***’, that is proportional to the energy lost per cycle.

NOTE: **Energy dissipation is usually determined under conditions of cyclic oscillations**.

ENERGY DISSIPATED IN VISCOUS DAMPING

Energy balance in a system with viscous damping **undergoing steady-state harmonic motion**:

Energy is input into the system by the applied force $p(t) = p_0 \sin(\Omega t)$. Then, the **energy input per cycle** is given by:

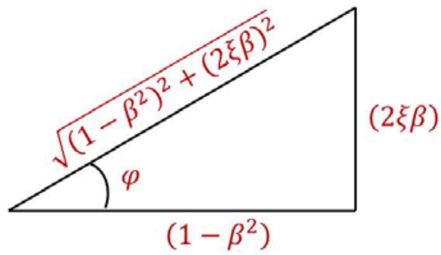
$$W_{input} = \oint p_0 \sin(\Omega t)\ du$$

But: $u(t) = \rho \sin(\Omega t - \varphi) \quad \Longrightarrow \quad du = \rho\Omega \cos(\Omega t - \varphi)\, dt$

Therefore: $W_{input} = \int_0^{2\pi/\Omega} p_0 \sin(\Omega t) \cdot \rho\Omega \cos(\Omega t - \varphi)\, dt =$ $p_0 \rho \pi \sin\varphi$

Recall that: $\tan\varphi = \frac{2\xi\beta}{1-\beta^2}$

$\Longrightarrow \quad \sin\varphi = \frac{2\xi\beta}{[(1-\beta^2)^2 + (2\xi\beta)^2]^{1/2}} = 2\xi\beta \frac{\rho}{\left(\frac{p_0}{k}\right)}$



Therefore:

$$W_{input} = 2\pi\xi\beta k\rho^2 \underbrace{=}_{\xi = \frac{c}{c_{cr}} \quad c_{cr} = \frac{2k}{\omega} \quad \beta = \frac{\Omega}{\omega}} c\pi\Omega\rho^2$$

The energy W_D dissipated by the **viscous damping force** $f_D = c\dot{u} = c\Omega\rho\cos(\Omega t - \varphi)$ is:

$$W_D = \oint f_D\, du = \int_0^{2\pi/\Omega} c\Omega\rho \cos(\Omega t - \varphi) \cdot \rho\Omega \cos(\Omega t - \varphi)\, dt = c\pi\Omega\rho^2$$

Therefore:

$$W_{input} = W_D = c\pi\Omega\rho^2$$

i.e., we demonstrated that **the total input is dissipated by viscous damping**.

The energy balance represented by the equation $W_{input} = W_D$ implies that **the work done by the spring and inertial forces per cycle is zero**.

This can also be proved by noting that:

$$f_S = ku = k\rho \sin(\Omega t - \varphi)$$

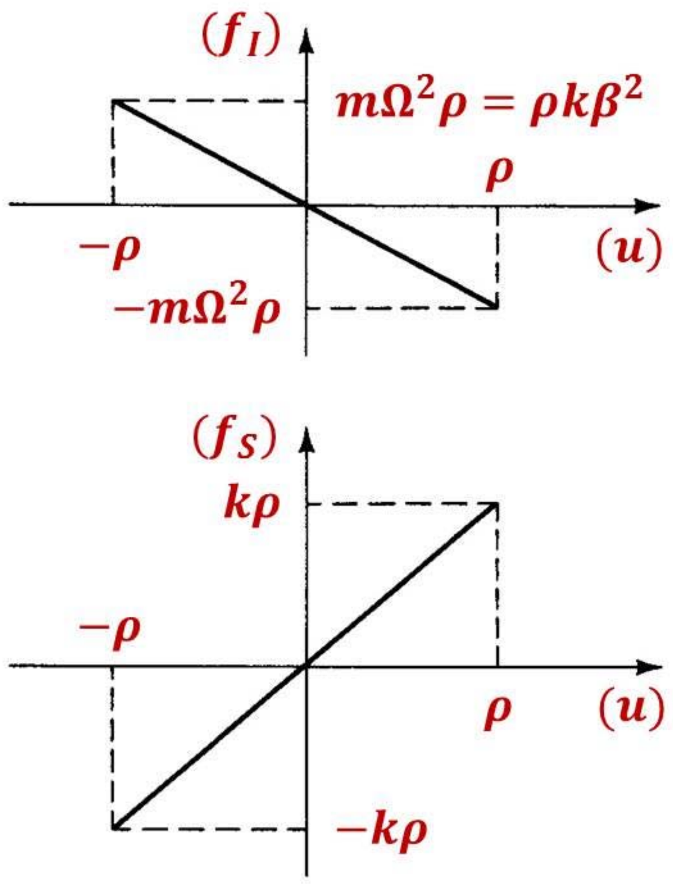
and
$$f_I = m\ddot{u} = -m\Omega^2 \rho \sin(\Omega t - \varphi) = -m\Omega^2 u$$

and integrating the infinitesimal work terms $f_S du$ & $f_I du$ over one cycle (see textbook).

A graphical method is probably more illustrative of the concepts involved:

The f_I vs. u relation (plotted below) consists of a single line as shown. Since the area enclosed by the f_I vs. u curve over one cycle of motion is zero, the corresponding work done should also be zero.

In a similar manner, the f_S vs. u diagram (drawn below) is a straight line and shows that the work done per cycle by f_S is also zero.



Since $\dot{u} = \Omega\rho\cos(\Omega t - \varphi) = \pm\Omega\rho\sqrt{1-\sin^2(\Omega t - \varphi)} = \pm\Omega\sqrt{\rho^2 - u^2}$

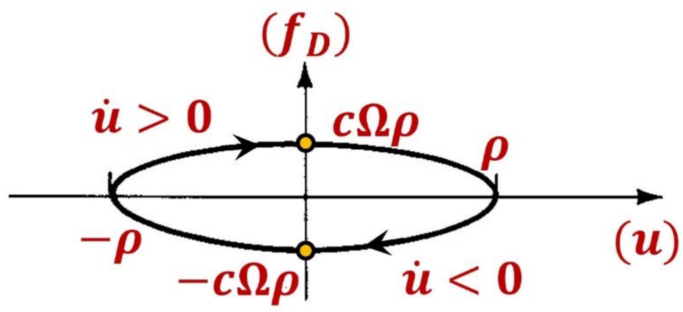
Therefore:

$$f_D = c\dot{u} = \pm c\Omega\sqrt{\rho^2 - u^2} \Rightarrow \boxed{\left(\frac{f_D}{c\Omega\rho}\right)^2 + \left(\frac{u}{\rho}\right)^2 = 1}$$

The above equation is the equation of an **ellipse** shown below. The **area of the ellipse** is

$$\boxed{\pi(c\Omega\rho)\rho = \pi c\Omega\rho^2}$$

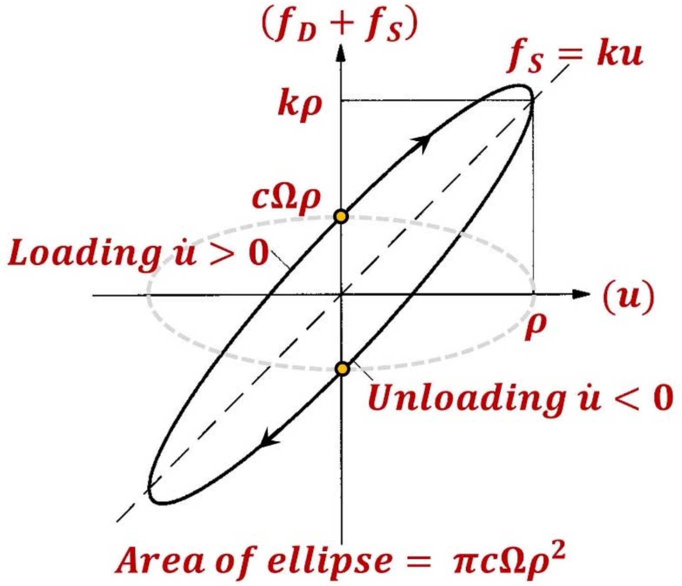
and represents the **energy dissipated per cycle in viscous damping**.



Total resisting force = $f_D + f_S$ (= **force measured in experiments**).

$$\begin{aligned} f_S + f_D &= ku + c\dot{u} \\ &= ku \pm c\Omega\sqrt{\rho^2 - u^2} \end{aligned}$$

A plot of $(f_D + f_S)$ vs. u is a **rotated ellipse**, as shown below (see APPENDIX below).

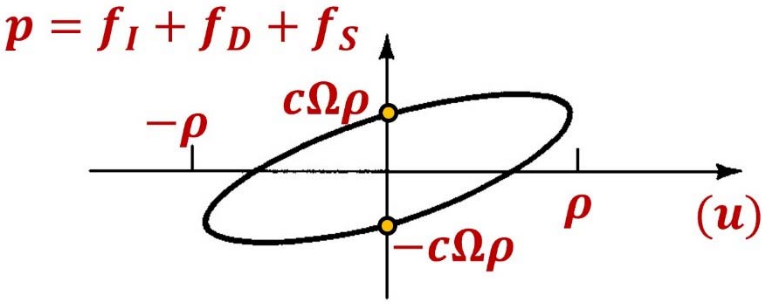


[Ellipse in Gray: f_D $vs.$ u]

Since $\boldsymbol{p(t) = f_I(t) + f_D(t) + f_S(t)}$, the applied force $\boldsymbol{p(t)}$ vs. displacement $\boldsymbol{u(t)}$ is given by:

$$p(t) = -\Omega^2 mu \pm c\Omega\sqrt{\rho^2 - u^2} + ku$$

The relationship $\boldsymbol{p(t)}$ vs. $\boldsymbol{u(t)}$ is a **rotated ellipse** (see APPENDIX below) whose principal axes are inclined *w.r.t.* to the coordinate axes. **The area of the skew ellipse** is equal to the area enclosed by the ellipse of $\boldsymbol{f_D(t)}$ vs. $\boldsymbol{u(t)}$, *i.e.*, equal to $(\boldsymbol{\pi c \Omega \rho^2})$ = **energy dissipated per cycle in viscous damping**.

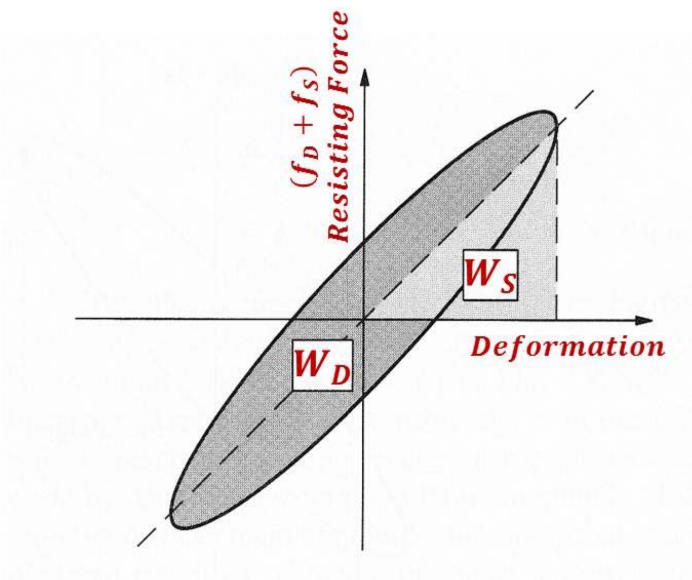


At **phase resonance** (*i.e.*, $\boldsymbol{\beta = 1}$) $\boldsymbol{f_D(t)}$ exactly balances $\boldsymbol{p(t)}$ (see corresponding **force balance diagram**) and the figure of $\boldsymbol{p(t)}$ vs. $\boldsymbol{u(t)}$ reduces (*i.e.*, becomes identical) to the figure of $\boldsymbol{f_D(t)}$ vs. $\boldsymbol{u(t)}$.

We mention two measures of damping:

$$\textit{Specific Damping Capacity} = \frac{W_D}{W_S} \qquad \left(where: W_S = \frac{1}{2}k\rho^2\right)$$

$$\textit{Loss Factor} = \frac{1}{2\pi}\frac{W_D}{W_S}$$



APPENDIX:

Consider the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Such an equation is **a general second-degree equation in rectangular coordinates** and always represents a ***conic*** (i.e., ellipse, parabola, hyperbola). The specific **type of conic** may be identified as follows (you may consult any book on 'Analytic Geometry', e.g., YATES, R.C., 1961. *Analytic Geometry with Calculus*, PRENTICE-HALL):

$$\begin{aligned} B^2 - 4AC < 0 &\Rightarrow \text{ellipse} \\ B^2 - 4AC = 0 &\Rightarrow \text{parabola} \\ B^2 - 4AC > 0 &\Rightarrow \text{hyperbola} \end{aligned}$$

If the conic is an ellipse, then its principal axes are rotated (w.r.t. the axes of the rectangular coordinates) by an angle ϑ, such that

$$\tan(2\vartheta) = \frac{B}{A-C}$$

As an example, we consider the total resisting $f_{res} \stackrel{\text{def}}{=} f_S + f_D$ vs. u for steady-state harmonic motion:

$$\begin{aligned} f_{res} &= ku \pm c\Omega\sqrt{\rho^2 - u^2} \\ \left(\frac{f_{res}}{k}\right) - u &= \pm\left(\frac{c\Omega}{k}\right)\sqrt{\rho^2 - u^2} \\ \left(\frac{f_{res}}{k}\right)^2 - 2\left(\frac{f_{res}}{k}\right)u + u^2 &= \left(\frac{c\Omega}{k}\right)^2(\rho^2 - u^2) \\ \left(1 + \left(\frac{c\Omega}{k}\right)^2\right)u^2 - 2\left(\frac{f_{res}}{k}\right)u + \left(\frac{f_{res}}{k}\right)^2 - \left(\frac{c\Omega\rho}{k}\right)^2 &= 0 \end{aligned}$$

Since

$$\left(\frac{c\Omega}{k}\right) = \frac{(2\xi\sqrt{km})\Omega}{k} = 2\xi\beta$$

then

$$(1 + (2\xi\beta)^2)u^2 - 2\left(\frac{f_{res}}{k}\right)u + \left(\frac{f_{res}}{k}\right)^2 - (2\xi\beta\rho)^2 = 0$$

Then, the expression $B^2 - 4AC$ for the above 2nd order equation becomes

$$2^2 - 4\cdot(1 + (2\xi\beta)^2)\cdot 1 = -4(2\xi\beta)^2 < 0$$

Therefore the 2nd order equation of $f_{res} \stackrel{\text{def}}{=} f_S + f_D$ vs. u represents an ellipse, the principal axes of which are rotated w.r.t. the axes (f_{res}/k) vs. u by an angle ϑ. Specifically,

$$\tan(2\vartheta) = \frac{-2}{(1 + (2\xi\beta)^2) - 1} = \frac{-2}{(2\xi\beta)^2}$$

Therefore,

$$\cos(2\vartheta) = -\frac{1}{\sqrt{1+(2/(2\xi\beta)^2)^2}} = -\frac{(2\xi\beta)^2}{\sqrt{(2\xi\beta)^4+4}}$$

$$\cos(2\vartheta) = 2\cos^2(\vartheta) - 1 \quad \Rightarrow \quad \cos(\vartheta) = \sqrt{\frac{1+\cos(2\vartheta)}{2}} = \sqrt{\frac{1}{2}\left(1-\frac{(2\xi\beta)^2}{\sqrt{(2\xi\beta)^4+4}}\right)}$$

Therefore

$$\vartheta = \cos^{-1}\sqrt{\frac{1}{2}\left(1-\frac{(2\xi\beta)^2}{\sqrt{(2\xi\beta)^4+4}}\right)}$$

A similar analysis may be performed to demonstrate that $p = f_I + f_D + f_S$ vs. u is an ellipse.

EQUIVALENT VISCOUS DAMPING

Definition of ***Equivalent Viscous Damping***:

DEFINITION #1:

Based on the measured response of the system at **phase resonance** (*i.e.*, $\boldsymbol{\beta = 1 \Leftrightarrow \Omega = \omega}$):

$$\xi = \frac{\left(\frac{p_0}{k}\right)}{2\rho}$$

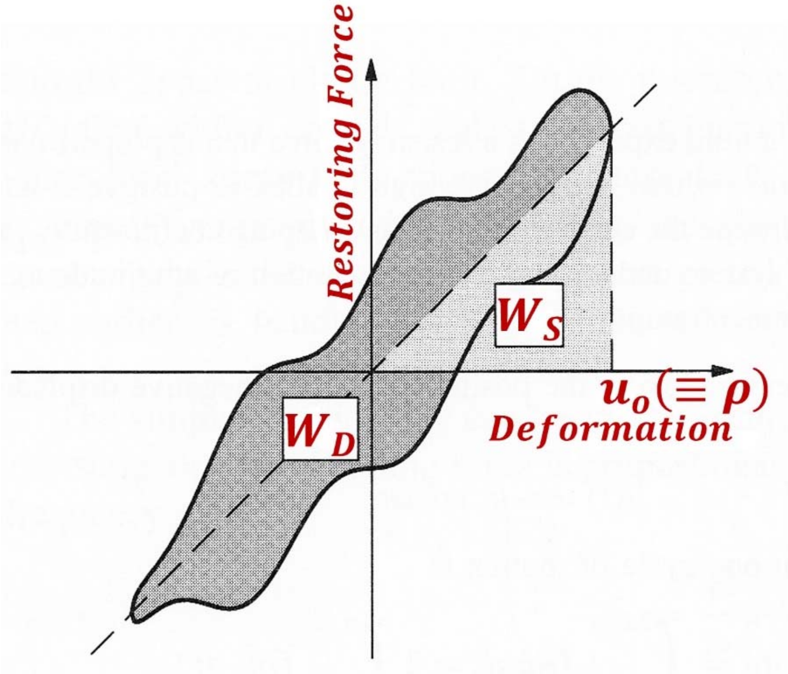
DEFINITION #2:

Equivalent viscous damping is the amount of damping that **provides the same bandwidth in the frequency-response curve as obtained experimentally** for an actual system:

$$\xi = \frac{\Omega_B - \Omega_A}{2\omega}$$

DEFINITION #3: (most common)

Equate the energy dissipated in a vibration cycle of the actual structure and an equivalent viscous system.



Let W_D = energy dissipated per cycle in the actual structure (i.e., area enclosed by the hysteresis loop)

$$\underbrace{4\pi\xi_{eq}\beta W_S = W_D}_{\textit{energy dissipated per cycle by equivalent viscous system}} \quad \Longrightarrow \quad \boxed{\xi_{eq} = \frac{1}{4\pi}\left(\frac{1}{\beta}\right)\frac{W_D}{W_S}}$$

W_S = strain energy $\left(\frac{1}{2}k\rho^2\right)$ calculated from the stiffness k determined by experiment

The experiment should be conducted at **phase resonance** (*i.e.*, $\beta = 1 \Leftrightarrow \Omega = \omega$), **where the response of the system is most sensitive to damping**, *i.e.*

$$\xi_{eq} = \frac{1}{4\pi}\frac{W_D}{W_S}$$

RATE-INDEPENDENT LINEAR DAMPING

[Also referred to as **HYSTERETIC** or **STRUCTURAL** or **SOLID DAMPING]**

Experiments indicate that **the energy dissipated per cycle** (and consequently corresponding force) **is independent of frequency** and **proportional to the square of the amplitude** of vibration (**unlike viscous damping** where the energy loss per cycle is proportional to the square of the amplitude and directly proportional to the frequency of motion).

This is because **damping forces are not viscous in nature** but **arise from internal friction**.

Rate-independent linear damping – Equation of Motion:

$$m\ddot{u} + f_S^t(u) = p(t)$$

where: $f_S^t(u)$ = nonlinear function of displacement

In the general case, the solution of the above equation is quite complex. However, **for steady-state harmonic motion**, rate-independent damping can be accounted for by expressing the total spring force $f_S^t(u)$ as the sum of two components: **an average spring force** $f_S = ku$, where k **is the average stiffness** and a damping force f_D given by

$$f_D = \left(\frac{\eta k}{\Omega}\right)\dot{u}$$

where η = a constant, indicating a dimensionless measure of damping.

Then, it is straightforward to demonstrate that **the energy dissipated per cycle of steady-state vibration at frequency Ω is independent of Ω**, *i.e.*,

$$W_D = \pi\eta k\rho_h^2 = 2\pi\eta W_S$$

Equation of motion: $$m\ddot{u} + f_S^t(u) = p(t)$$

Total Spring/Resisting Force: $$f_S^t(u) = \underbrace{ku}_{f_S} + \underbrace{\frac{\eta k}{\Omega}\dot{u}}_{f_D}$$

It follows that: $$m\ddot{u} + \frac{\eta k}{\Omega}\dot{u} + ku = p_0 \sin(\Omega t)$$

We define: $$c_h = \frac{\eta k}{\Omega} \quad , \quad \xi_h = \frac{c_h}{(2\sqrt{km})} = \frac{\eta}{(2\beta)}$$

Clearly, while $\boldsymbol{\eta} = const.$, $\boldsymbol{c_h}$ & $\boldsymbol{\xi_h}$ vary with $\boldsymbol{\beta} = \left(\frac{\Omega}{\omega}\right)$.

Then, **the steady-state response** is:

$$u_{ss}(t) = \rho_h \sin(\Omega t - \varphi_h)$$

where:

$$\rho_h = \frac{p_0}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + \eta^2}}$$

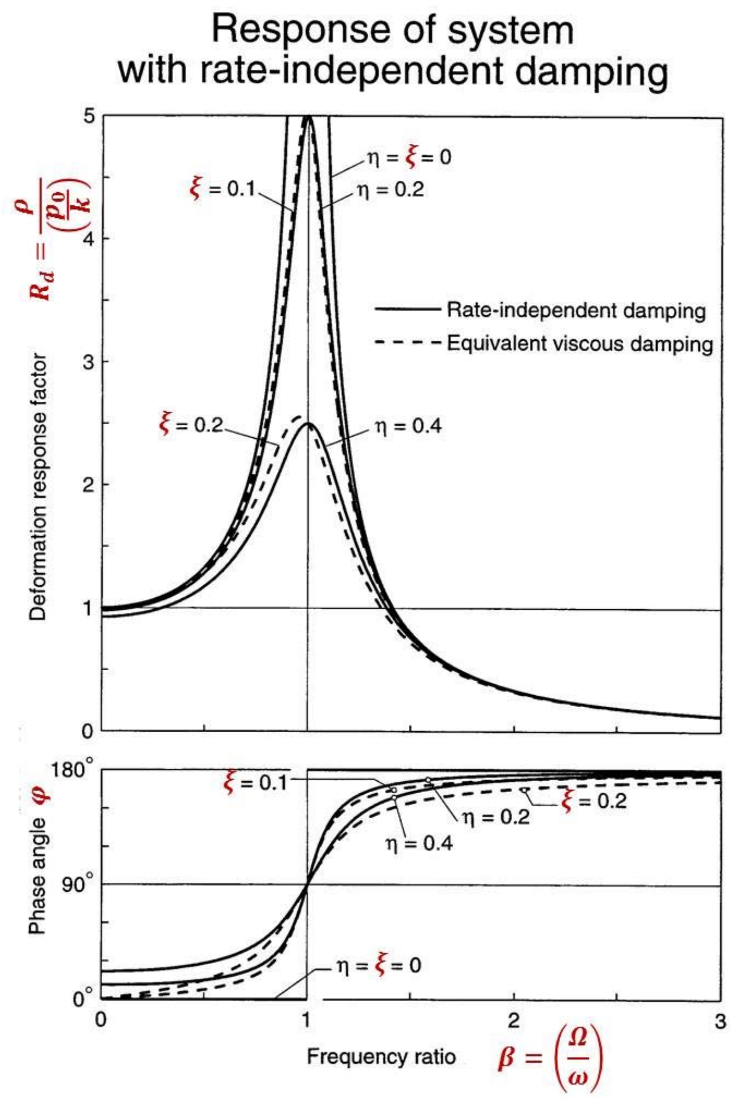
$$\tan \varphi_h = \frac{\eta}{1-\beta^2}$$

Approximate Solution Using Equivalent Viscous Damping:

Matching dissipated energies at phase resonance ($\boldsymbol{\beta = 1 \Leftrightarrow \Omega = \omega}$), we obtain:

$$\xi_{eq} = \frac{\eta}{2}$$

- The approximate solution matches the exact results at $\boldsymbol{\Omega = \omega}$ because that was the criterion used in selecting $\boldsymbol{\xi_{eq}}$.
- **Amplitude resonance** (*i.e.*, **maximum amplitude**) **for the exact solution occurs at** $\boldsymbol{\Omega = \omega}$, while **for the approximate solution it occurs at** $\boldsymbol{\Omega < \omega}$.
- The phase angle $\boldsymbol{\varphi_h}$ for $\boldsymbol{\Omega = 0}$ is $\boldsymbol{\varphi_h = \tan^{-1}\eta}$ for the exact solution. For the approximate solution, the phase angle is zero for $\boldsymbol{\Omega = 0}$. This implies that **motion with rate independent damping can never be in phase with the forcing function**.



Complex Stiffness

For simple harmonic motion, hysteretic damping can conveniently be expressed by using the concept of ***complex stiffness***.

$$m\ddot{u} + \left(\frac{\eta k}{\Omega}\right)\dot{u} + ku = p_0 e^{i\Omega t}$$

Response:

$$\begin{aligned} u &= U e^{i\Omega t} \\ \dot{u} &= U(i\Omega)e^{i\Omega t} = (i\Omega)u \end{aligned}$$

Damping force:

$$\begin{aligned} f_D &= \left(\frac{\eta k}{\Omega}\right)\dot{u} \\ &= \left(\frac{\eta k}{\Omega}\right)(i\Omega)u \\ &= i\eta k u \end{aligned}$$

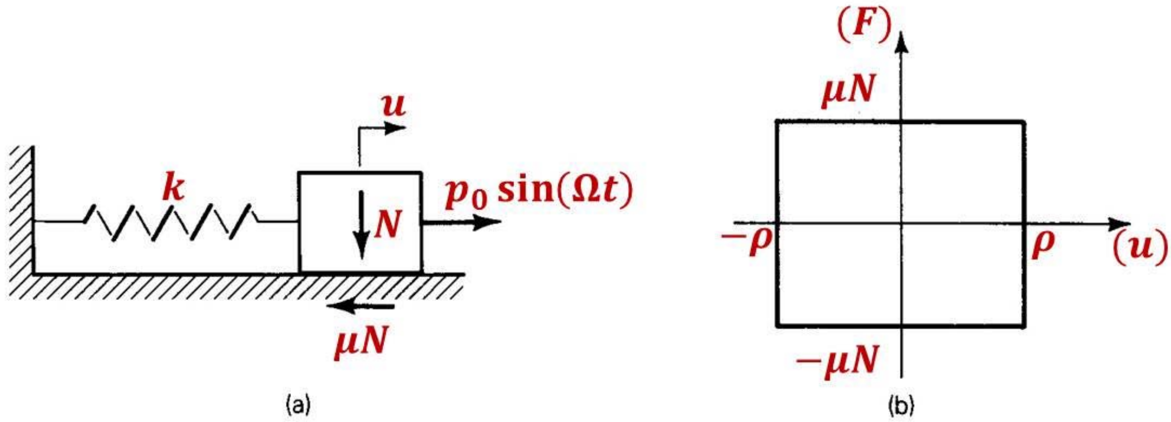
Therefore:

$$m\ddot{u} + \underbrace{(1 + i\eta)k}_{\bar{k}}\, u = p_0 e^{i\Omega t}$$

$$\boxed{\bar{k} = (1 + i\eta)k \quad \textit{Complex Stiffness}}$$

The concept of complex stiffness applies **only** for **harmonic oscillations**, and it is useful in taking account of rate-independent (*i.e.*, hysteretic) damping **when the analysis is carried out in the frequency domain**.

HARMONIC VIBRATION WITH COULOMB FRICTION



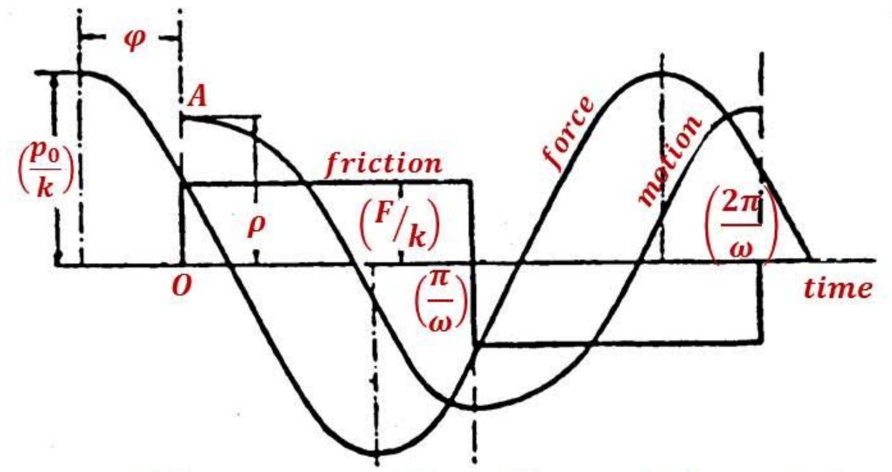
Equation of Motion:

$$m\ddot{u} + ku + \text{sgn}(\dot{u})\underbrace{(\mu N)}_{F} = p_0 \sin(\Omega t)$$

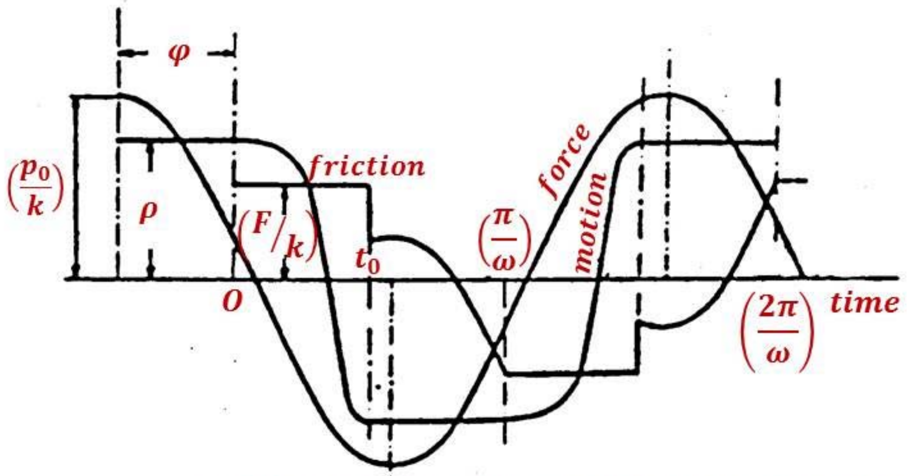
Even though the differential equation is linear over each time segment that $\dot{u}(t)$ does not change sign (*i.e.*, over each time segment that the block moves in the same direction) **the problem is nonlinear**.

An **exact analytical solution for the steady-state response** of the system subjected to harmonic force was developed by **J.P. DEN HARTOG** (**1930, 1931**).

NOTE: It must be remembered that **the friction force is passive one**, which means that **it is able to oppose the motion but not to produce it**.



CASE I*: *Diagram of motion without stops



CASE II*: *Diagram of motion with stops

NOTE: For all cases the **response** of the system is **periodic** but **non-harmonic**.

After the system has been in motion for some time a '**steady-state**' will be reached which **must satisfy** the following conditions:

1) **The frequency of the motion must be the same as that of the disturbing force.**
2) **The downward half-cycle of the motion must follow the same law as the upward half-cycle.**

Cases (I) & (II) above show two possible types of such motion.

CASE (I):

When the friction force $\boldsymbol{F}$ is sufficiently smaller than $\boldsymbol{p_0}$, a continuous motion (i.e., the block/mass never comes to a dead stop) will occur and a type of steady-state solution will result.

For this type of motion the maximum displacement $\boldsymbol{\rho}$ is given by:

$$\rho = \frac{p_0}{k}\sqrt{V^2 - \left(\frac{F}{p_0}\right)^2 U^2}$$

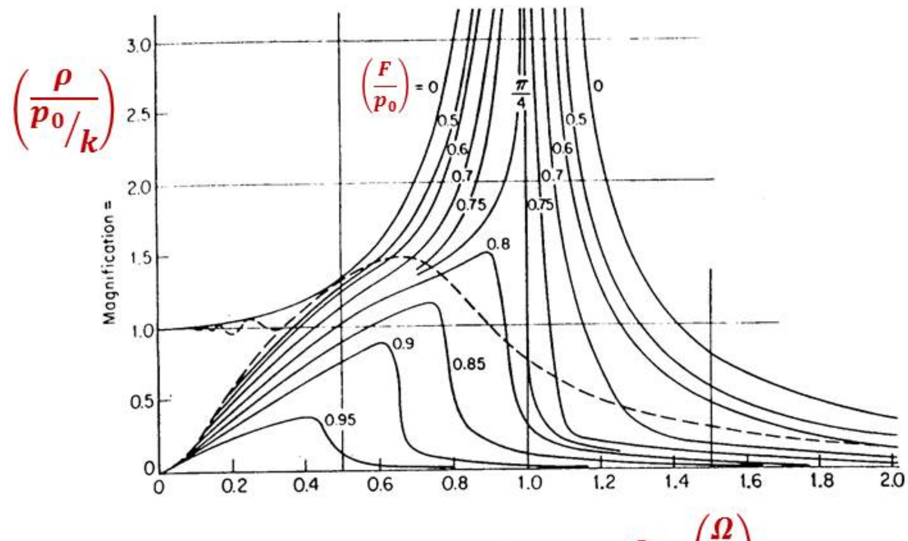
$$\text{where:} \quad \begin{cases} V = \dfrac{1}{1-\beta^2} \\ U = \dfrac{1}{\beta}\tan\left(\dfrac{\pi}{2\beta}\right) \end{cases}$$

In the attached $[\boldsymbol{\rho}/(\boldsymbol{p_0}/\boldsymbol{k})]$ vs. $\boldsymbol{\beta}$ FIGURE, **only the part of the diagram above the broken line corresponds to the above formula**.

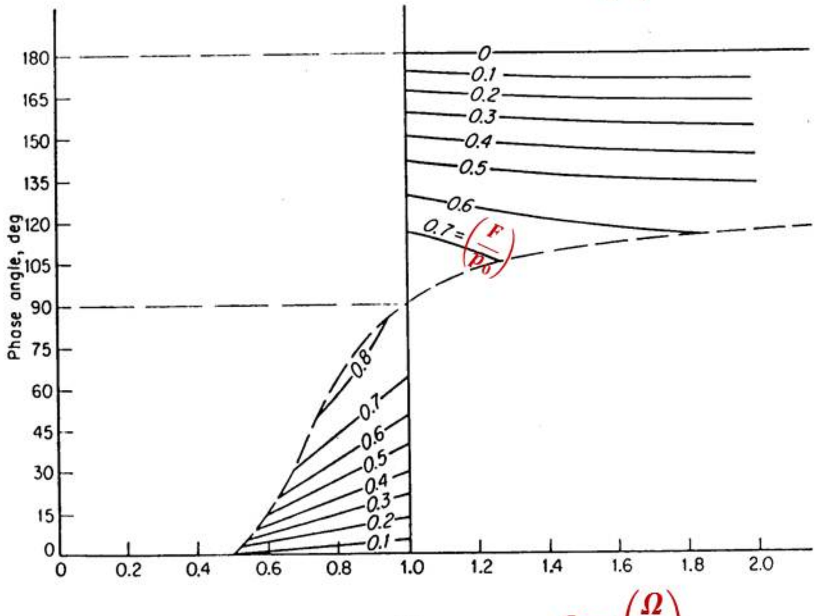
Notice that:

- $\boldsymbol{V}$ is the **deformation response function** of the un-damped SDOF system.
- If $\boldsymbol{F = 0}$, the above expression for $\boldsymbol{\rho}$ reduces to the well known one for the un-damped SDOF system.
- **At the upper and lower peaks, the displacement curve suddenly changes its curvature** (see attached FIGURE, CASE I). The explanation of this is that the curvature is proportional to the second derivative. The second derivative of the displacement is the acceleration, which is proportional to all the forces acting on the mass. Since the frictional force suddenly reverses at the peaks, the curvature there must show a discontinuous change.

CASE (I): (continued)



(a)



(b)

CASE (I): (continued)

Phase Resonance $\boldsymbol{\beta = 1 \Leftrightarrow \Omega = \omega}$:

A remarkable property of the above response/solution is that for

$$\frac{F}{p_0} < \frac{\pi}{4} (= 0.785)$$

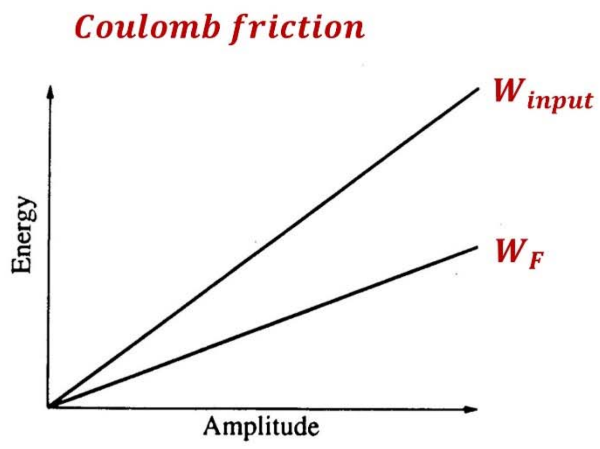
The response amplitude becomes infinitely large. In other words, **even with considerable amount of Coulomb friction, the amplitudes at resonance still extend to infinity**. This apparently paradoxical fact can be understood by considering the facts involved as follows:

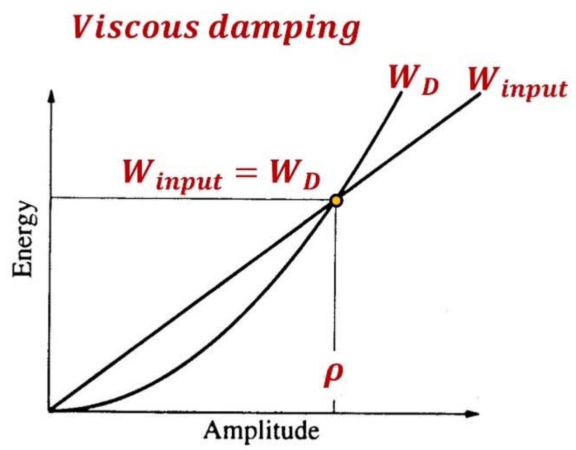
It can be demonstrated that **at resonance $(\boldsymbol{\beta = 1})$, and assuming that the condition $\left(\frac{F}{p_0}\right) < \left(\frac{\pi}{4}\right)$** is satisfied, the following relation is true:

$$W_F < W_{input}$$

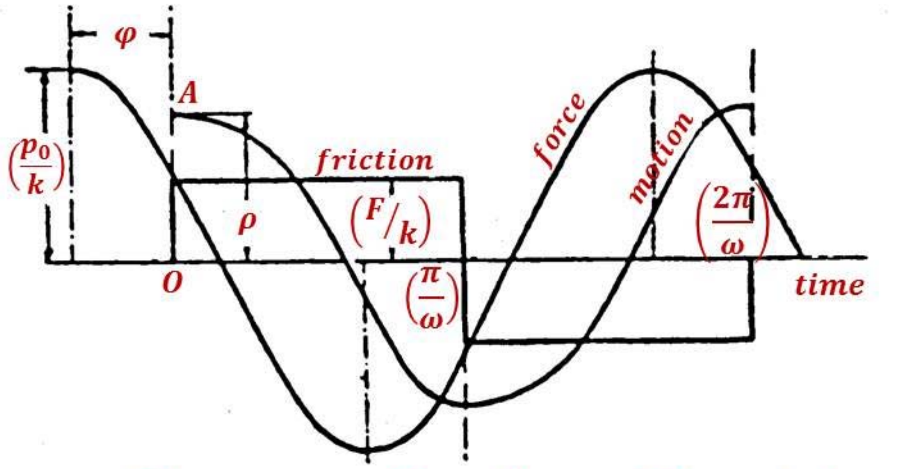
That is, **the energy dissipated in friction per cycle is <u>less</u> than the input energy**. Therefore, the displacement amplitude would increase cycle after cycle and grow without bound.

The above behavior is quite different from that of a system with viscous damping, or **rate-independent damping**. For these forms of damping, $\boldsymbol{W_D = c\pi\Omega\rho^2}$ or $\boldsymbol{W_D = \pi\eta k\rho_h^2}$ while the input energy is $\boldsymbol{W_{input} = p_0\rho\pi\sin\left(\varphi = \frac{\pi}{2}\right) = p_0\rho\pi}$. Thus, in these cases, $\boldsymbol{W_D}$ & $\boldsymbol{W_{input}}$ match each other at the steady-state amplitude $\boldsymbol{\rho}$, which will be bounded no matter how small the damping.

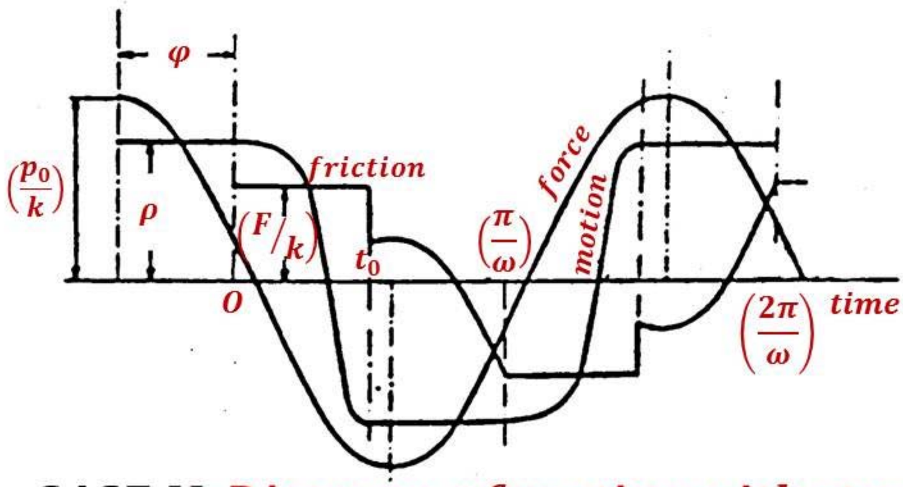




CASE (II):



CASE I*: *Diagram of motion without stops

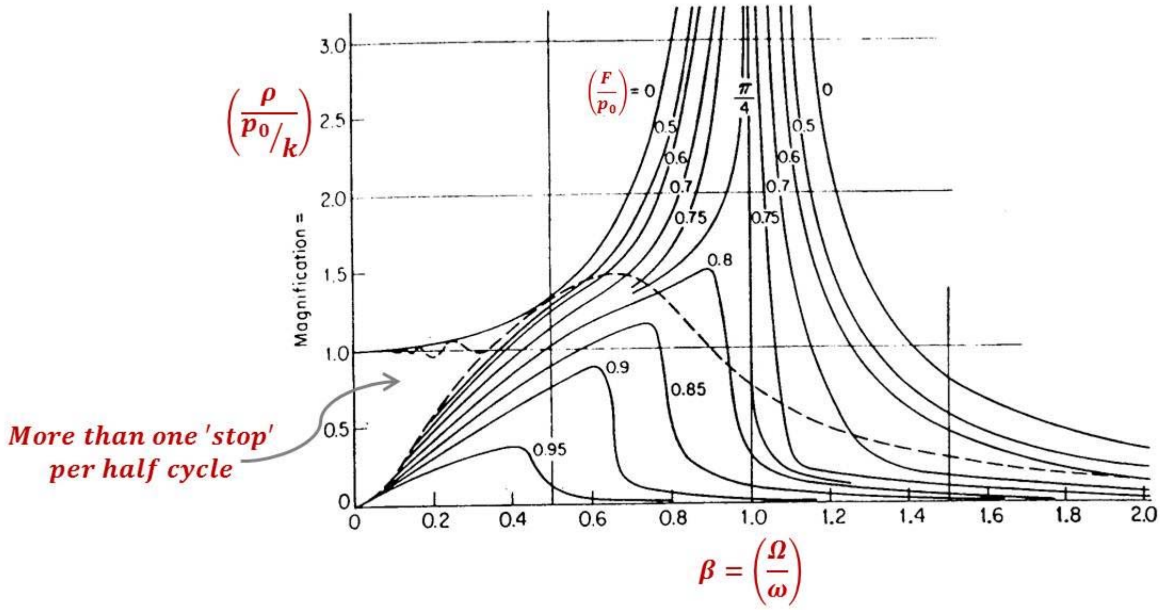


CASE II*: *Diagram of motion with stops

An increase in the friction force F will decrease the amplitude of motion and will also reduce the curvature at point A (see FIGURE corresponding to CASE I above). When F reaches a certain value, the curvature becomes zero, and for larger values of F than this, the motion of FIGURE/CASE (I) above cannot exist and will be replaced by that of FIGURE/CASE (II). During each half-cycle the mass m will stop moving for a while, and while stopping the value of the friction force may be anything between $+F$ and $-F$. Since the mass m during this interval has no acceleration, then it follows that $ku - \mathrm{sgn}(\dot{u})F = p(t)$. This last relation determines the friction force F as shown in FIGURE/CASE (II).

CASE (II): (continued)

NOTE: For **CASE (II)** (*i.e.*, **one 'stop' per half cycle**) it is possible to obtain an **exact solution**, but it is **not possible to put it into explicit form**.

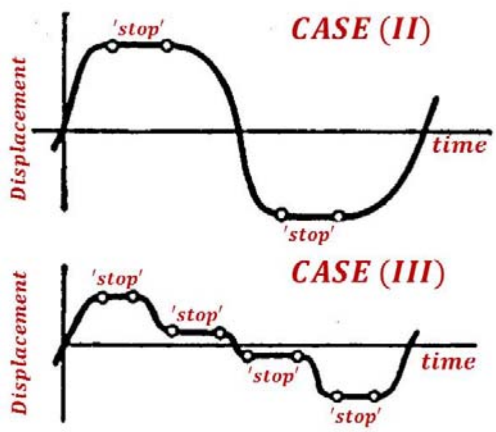


For **Coulomb friction**, the **friction force *F*** is **constant**, whereas f_I and f_S increase with amplitude of displacement.

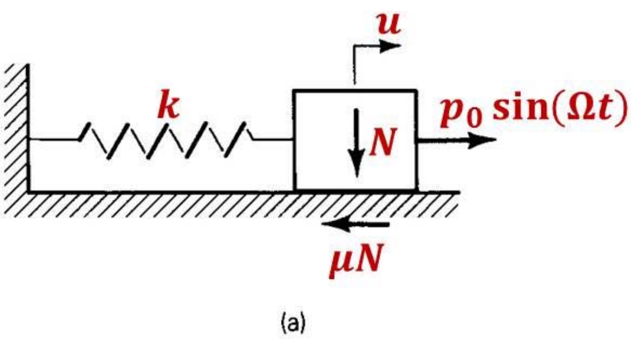
Thus, **for large amplitudes the motion will be practically sinusoidal and the 'equivalent viscous damping' approximation should be very satisfactory. For smaller amplitudes the curve of motion becomes very distorted and consequently the 'equivalent viscous damping' approximation for the amplitude is poor.**

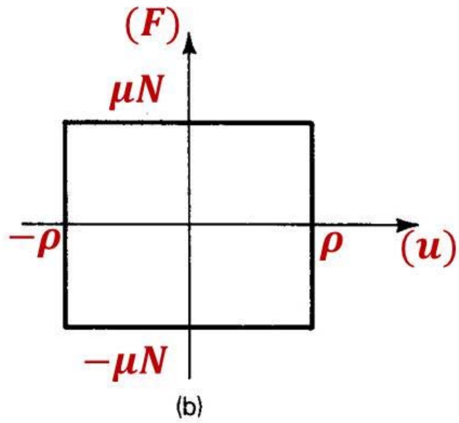
Below the intermittent line running through the FIGURE, of the deformation response function (above), **we have motions with one 'stop' per half cycle**.

In the blank part in the left half lower corner of the above FIGURE, the motion has **more than one stop per half-cycle** (see **CASE III** above). **No solution could be obtained in that region**. In practice, however, we are interested only in the conditions near resonance.



COULOMB DAMPING: *Solution Using Equivalent Viscous Damping*





Equation of Motion:

$$m\ddot{u} + ku + \text{sgn}(\dot{u})\underbrace{(\mu N)}_{F} = p_0 \sin(\Omega t)$$

The work done by the friction force per cycle of **steady-state** motion is:

$$W_F = 4F\rho = 4(\mu N)\rho$$

Equivalent Viscous Damping:

$$\underbrace{\rho^2 \pi \Omega c_{eq}}_{\text{energy loss per cycle by viscous damping}} = \underbrace{4F\rho}_{\text{energy loss per cycle by Coulomb friction}}$$

It follows that:

$$c_{eq} = \frac{4F}{\rho \pi \Omega}$$

Therefore:

$$\xi_{eq} = \frac{c_{eq}}{2\sqrt{km}} = \frac{2F}{\pi k \rho \beta}$$

Displacement response amplitude:

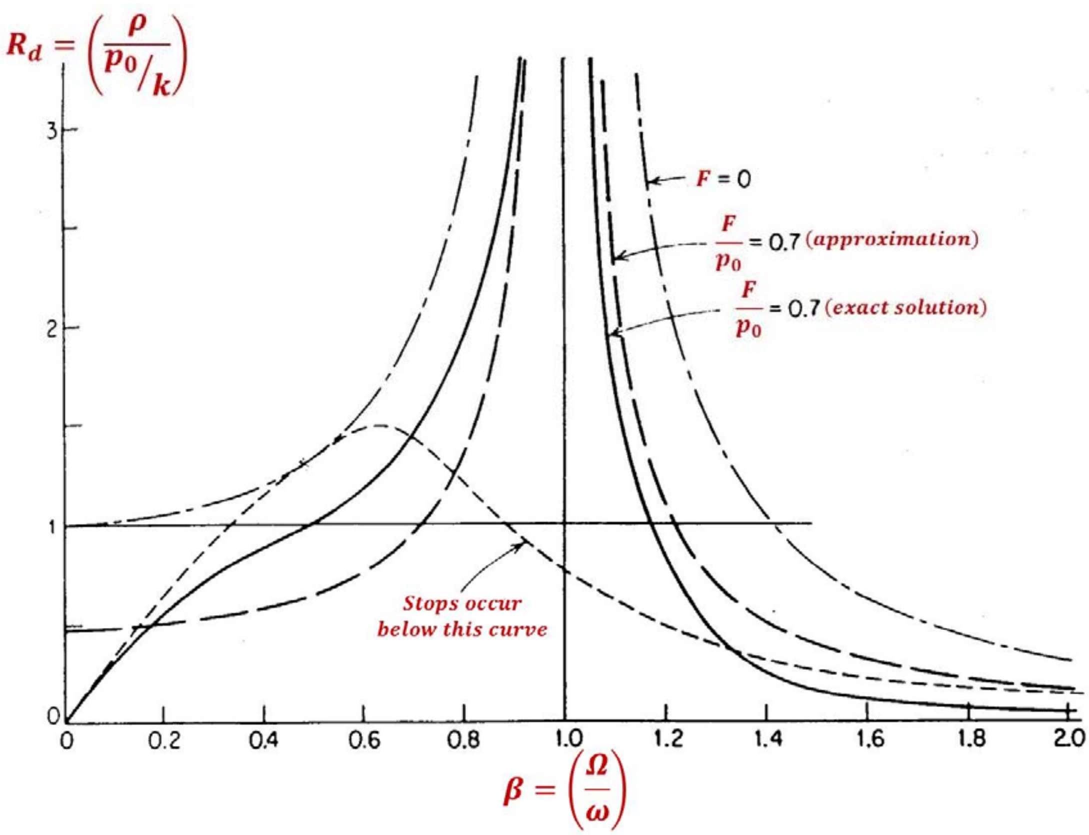
$$\rho = \left(\frac{p_0}{k}\right)\frac{1}{\sqrt{(1-\beta^2)^2 + \left(\frac{4F}{\pi k \rho}\right)^2}}$$

The above expression contains ρ on the right hand side also. Solving for ρ, we obtain:

$$\boxed{\rho = \left(\frac{p_0}{k}\right)\frac{\sqrt{1-\left(\frac{4F}{\pi p_0}\right)^2}}{(1-\beta^2)}}$$

The above result is valid, provided that $\left(\frac{F}{p_0}\right) < \frac{\pi}{4}(= 0.785)$. For larger friction force, the numerator becomes imaginary and the method breaks down. In fact, **for reasonable** accuracy, $\left({}^{F}/_{p_0}\right)$ **must be less than about** $\left({}^{1}/_{2}\right)$.

The FIGURE below shows a comparison of **the exact solution** (**DEN HARTOG, 1931**) with **the approximate one** (*i.e.*, **equivalent viscous damping**) for $\left({}^{F}/_{p_0}\right) = 0.7$. The general conclusion is that the approximate solution applied to the constant-friction system underestimates the amplitude below resonance, while above resonance the values it gives are too large. In the resonance region the agreement is reasonably good.



The steady-state response of the equivalent viscous system (*i.e.*, the approximation) is represented as follows:

$$u_{ss}(t) = \rho \sin(\Omega t - \varphi)$$

$$\text{where: } \begin{cases} \rho = \left(\dfrac{p_0}{k}\right)\dfrac{\sqrt{1-\left(\dfrac{4F}{\pi p_0}\right)^2}}{(1-\beta^2)} \\ \tan\varphi = \pm\dfrac{\left(\dfrac{4F}{\pi p_0}\right)}{\sqrt{1-\left(\dfrac{4F}{\pi p_0}\right)^2}} \end{cases}$$

For the **phase** φ $\begin{cases} \text{the } (+) \text{ sign applies for } \beta < 1 \\ \text{the } (-) \text{ sign applies for } \beta > 1 \end{cases}$

The phase is independent of β but it changes sign as β passes through **1**.