MEASUREMENT OF DAMPING

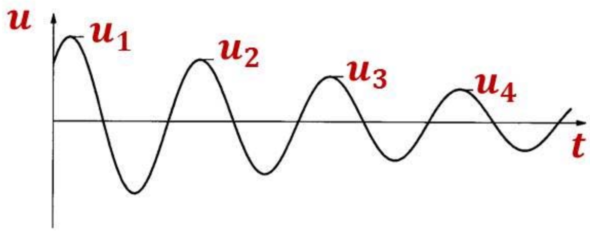
Mass and **stiffness** of a dynamic system can be determined by its physical characteristics, while an estimate of **damping** resistance can be obtained by experimental measurements of the response of the structure to a given excitation.

Experimental Evaluation Techniques of Damping Ratios:

- *Free Vibration Decay*

- *Forced-Vibration Response:*
    - Resonant Response
    - **Width** of Response Curve & **Half-Power Method**
    - Energy Loss per Cycle

FREE VIBRATION DECAY – LOGARITHMIC DECREMENT:



Free vibration of an under-damped system:

Let:

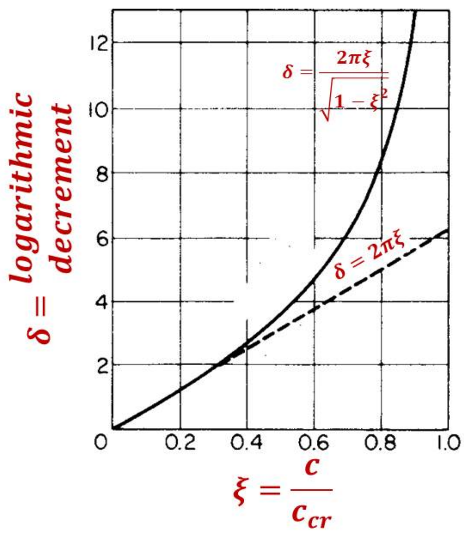
$$u_1 \stackrel{\text{def}}{=} u(t_1) = \rho e^{-\xi\omega t_1}\sin(\omega_d t_1 + \varphi)$$

Then:

$$u\left(t_1 + \frac{2\pi}{\omega_d}\right) = \rho e^{-\xi\omega\left(t_1+\frac{2\pi}{\omega_d}\right)} \underbrace{\sin\left[\omega_d\left(t_1 + \frac{2\pi}{\omega_d}\right) + \varphi\right]}_{\sin(\omega_d t_1+\varphi)}$$

The ratio $u(t_1)$ to $u\left(t_1 + \frac{2\pi}{\omega_d}\right)$ provides a **measure of the decrease in displacement over one cycle** of motion. The **ratio is constant and does not vary with time**; its natural logarithm is called ***logarithmic decrement*** and is denoted by $\boldsymbol{\delta}$.

[**NOTE**: The idea is due to **HELMHOLTZ, 1862**.]



$$\begin{aligned}\delta &= \ln\left\{\frac{u(t_1)}{u\left(t_1+\frac{2\pi}{\omega_d}\right)}\right\} \\ &= \ln\left\{\frac{e^{-\xi\omega t_1}}{e^{-\xi\omega\left(t_1+\frac{2\pi}{\omega_d}\right)}}\right\} \\ &= 2\pi\xi\left(\frac{\omega}{\omega_d}\right) \\ &= 2\pi\frac{\xi}{\sqrt{1-\xi^2}}\end{aligned}$$

Therefore:

$$\boxed{\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}}$$

[For small values of ξ, *i.e.*, $\boldsymbol{\xi \ll 1} \Longrightarrow \delta \cong 2\pi\xi \Longleftrightarrow \boldsymbol{\xi \cong \frac{\delta}{2\pi}}$]

If the **decay** of motion is **slow**, it is desirable to **relate the ratio of two amplitudes several cycles apart**, instead of successive amplitudes, to the damping ratio.

$$\frac{u_1}{u_{n+1}} = \frac{u_1}{u_2}\cdot\frac{u_2}{u_3}\cdot\frac{u_3}{u_4}\cdots\frac{u_n}{u_{n+1}} = e^{n\delta}$$

Therefore:

$$\delta_n \overset{\text{def}}{=} \ln\left(\frac{u_1}{u_{n+1}}\right) = n\delta = n\cdot 2\pi\frac{\xi}{\sqrt{1-\xi^2}}$$

It follows that:

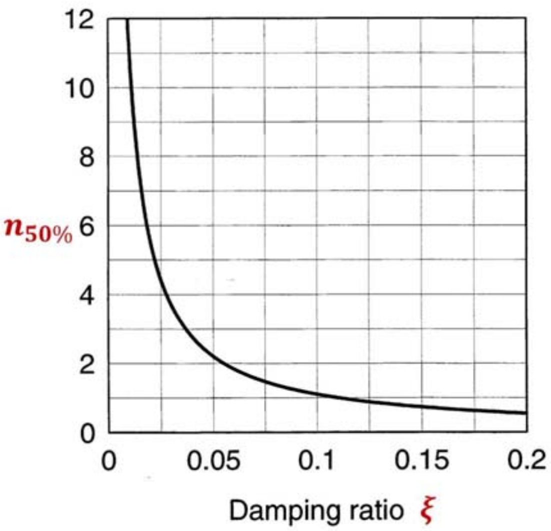
$$\boxed{\xi = \frac{\delta_n}{\sqrt{(2\pi n)^2+\delta_n^2}}}$$

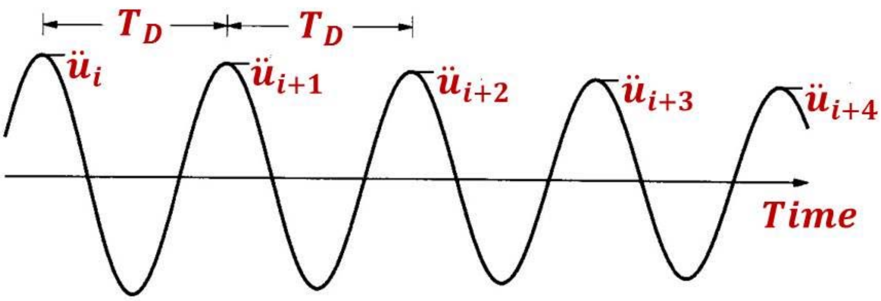
For **lightly damped** systems (*i.e.*, **small** ξ):

$$\delta_n = n\delta \cong n2\pi\xi \implies \boxed{\xi \cong \frac{\delta_n}{2\pi n}}$$

To determine the **number of cycles** elapsed for a 50% reduction in displacement amplitude, we obtain:

$$n_{50\%} \cong \frac{\ln 2}{2\pi\xi} = \frac{0.11}{\xi}$$





The damping ratio ξ can be obtained from:

$$\xi = \frac{1}{2\pi n} ln\left(\frac{u_i}{u_{i+n}}\right)$$

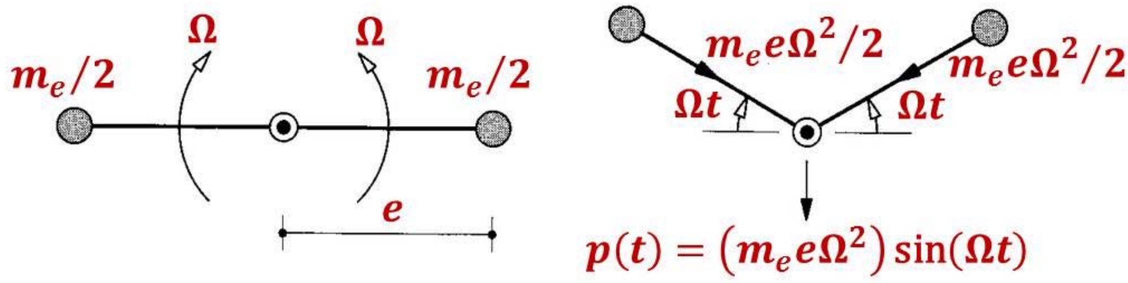
or

$$\xi = \frac{1}{2\pi n} ln\left(\frac{\ddot{u}_i}{\ddot{u}_{i+n}}\right)$$

Accelerations are easier to measure than displacement. [The latter equation (i.e. the equation involving accelerations) is (approximately) **valid for lightly damped systems**.]

The **damped period** $T_d = T/\sqrt{1-\xi^2}$ of the system can be **determined from the free vibration record** by measuring the time required to complete one cycle of vibration.

RESPONSE TO VIBRATION GENERATOR



Counter-rotating eccentric-weight vibration generator

Consider **two counter-rotating masses**, $m_e/2$.

Harmonic force produced: $p(t) = (m_e e\Omega^2)\sin(\Omega t)$

Equation of motion (assuming $m_e \ll m$ = mass of structure):

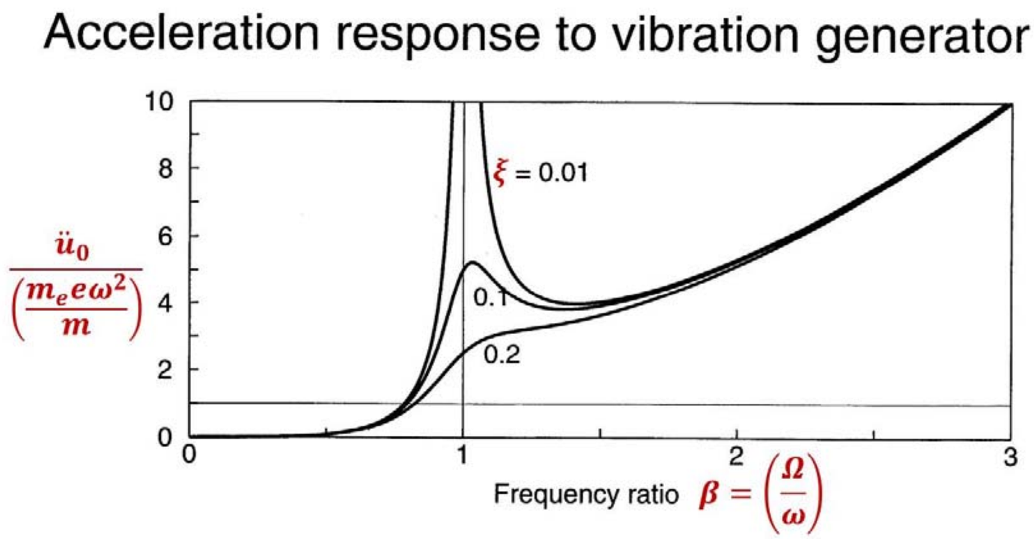
$$m\ddot{u} + c\dot{u} + ku = \underbrace{(m_e e\Omega^2)}_{p_0}\sin(\Omega t) \quad \Rightarrow \quad u_{ss}(t) = \frac{(m_e e\Omega^2)}{k} R_d \sin(\Omega t - \varphi)$$

Therefore:

$$\rho = \frac{(m_e e\Omega^2)}{k} \cdot R_d = \frac{m_e e}{m} \cdot (\beta^2 R_d) = \frac{m_e e}{m} \cdot R_a$$

$$\ddot{u}_0 = \Omega^2 \rho = \frac{m_e e\omega^2}{m} \cdot (\beta^2 R_a)$$

$$\boxed{R_d \quad , \quad R_v = \beta R_d \quad , \quad R_a = \beta^2 R_d}$$



FORCED-VIBRATION RESPONSE

Resonant Response:

At **phase resonance**, $\beta = 1$ & phase angle $\varphi = \frac{\pi}{2}$

Therefore, **phase resonance can be detected by measuring the phase angle** and progressively adjusting the exciting frequency until $\varphi = \frac{\pi}{2}$.

$$R_d|_{\beta=1} = \frac{\rho_{\beta=1}}{\left(\frac{p_0}{k}\right)} = \frac{1}{2\xi} \quad \Longrightarrow \quad \boxed{\xi = \frac{1}{2}\frac{\left(\frac{p_0}{k}\right)}{\rho_{\beta=1}}}$$

Usually, **the acceleration amplitude $\ddot{u}_0$ is measured** and

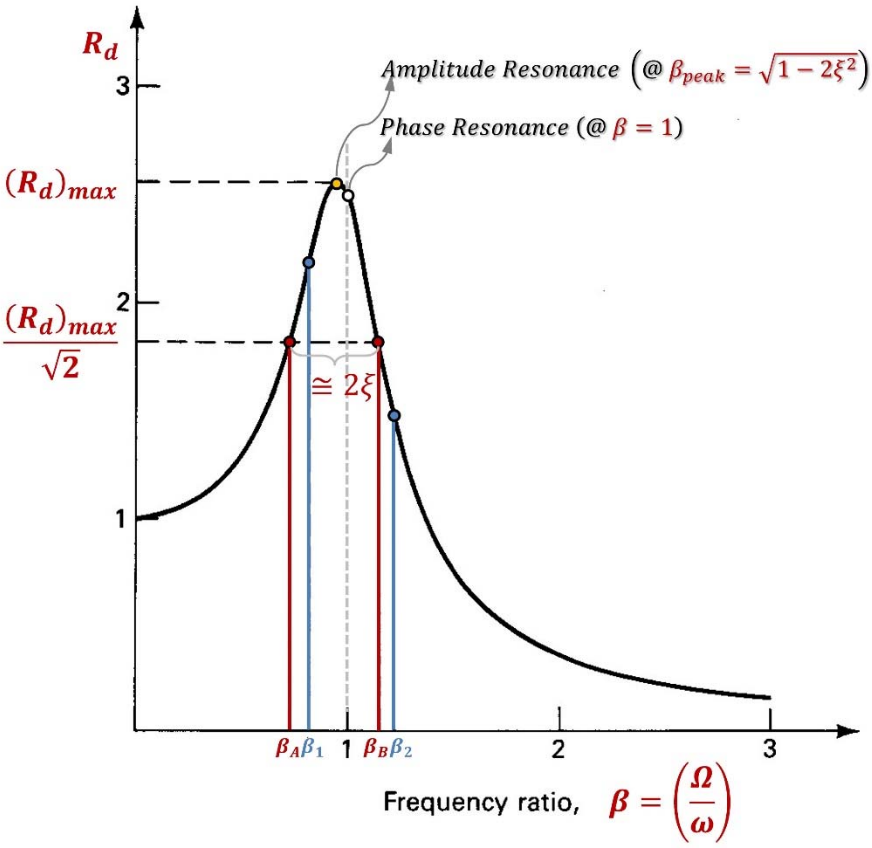
$$\boxed{\rho = \frac{\ddot{u}_0}{\Omega^2}}$$

Measurement of the phase angle may be somewhat difficult. Therefore, as an alternative the resonance curve is obtained in the vicinity of resonance and the peak response $\boldsymbol{\rho_{max}}$ is measured. For viscous damping

$$(R_d)_{max} = \frac{\rho_{max}}{\left(\frac{p_0}{k}\right)} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

The above equation can be used to obtain $\boldsymbol{\xi}$ from the measured value of $\boldsymbol{\rho_{max}}$. For light damping $\xi \cong \frac{1}{2}\frac{(p_0/k)}{\rho_{max}}$.

Width of Response Curve Method:



The width of response curve near resonance can be used to obtain an estimate of the damping.

Measurement of frequencies Ω_1 & Ω_2 at which $\varphi = \pm\left(\frac{\pi}{4}\right) \Longrightarrow \tan\varphi = \pm 1$

Therefore:

$$\frac{2\xi\beta_1}{1-\beta_1^2} = 1 \ \ \& \ \ \frac{2\xi\beta_2}{1-\beta_2^2} = -1 \ \Longrightarrow$$

$$\Longrightarrow \left.\begin{matrix} 1-\beta_1^2 - 2\xi\beta_1 = 0 \\ 1-\beta_2^2 + 2\xi\beta_2 = 0 \end{matrix}\right\} \Longrightarrow$$

$$\Longrightarrow \boxed{\xi = \frac{1}{2}(\beta_2 - \beta_1) = \frac{1}{2}\left(\frac{\Omega_2 - \Omega_1}{\omega}\right)}$$

The above method **relies on the ability to measure the phase angle**, which **may require sophisticated instrumentation**.

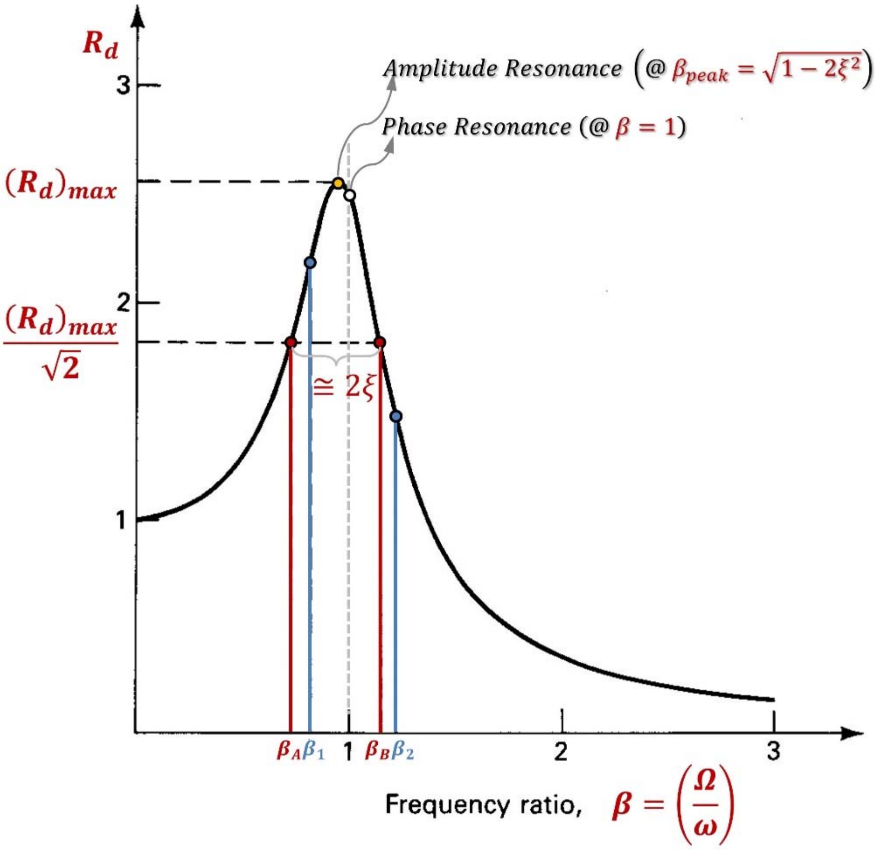
Half-Power Method:

NOTE:

The **ratio**

$$Q = \frac{\omega}{\Omega_B - \Omega_A} \cong \frac{1}{2\xi}$$

is referred to as the ***quality factor*** and is a **measure of the sharpness of the response curve**.



If the response curve in the vicinity of resonance has been plotted, the frequencies at which the amplitude is $(\mathbf{1}/\sqrt{\mathbf{2}})$ times that at the peak can be measured. As shown in the above FIGURE there are two such frequencies, denoted Ω_A & Ω_B and the corresponding ratios β_A & β_B. [**The power ($\sim\rho^2$) at $\beta_{A,B}$ is half the power at β_{peak}, hence the name *Half-Power Method*.**]

$$R_d(\beta_{A,B}) = \frac{1}{\sqrt{2}} R_d(\beta_{peak}) \quad \Longrightarrow \frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}} = \frac{1}{\sqrt{2}} \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\Longrightarrow \beta^4 - 2(1-2\xi^2)\beta^2 + 1 - 8\xi^2(1-\xi^2) = 0$$

$$\Longrightarrow \beta^2 = (1-2\xi^2) \pm 2\xi\sqrt{1-\xi^2}$$

For small damping $\quad \Longrightarrow \beta^2 = \left(\frac{\Omega}{\omega}\right)^2 \cong 1 \pm 2\xi \qquad \therefore \beta = \left(\frac{\Omega}{\omega}\right) \cong \sqrt{1 \pm 2\xi} \cong 1 \pm \xi$

Therefore:

$$\frac{\Omega_B - \Omega_A}{\omega} \cong 2\xi \quad \Longrightarrow \quad \boxed{\xi = \frac{1}{2}(\beta_B - \beta_A) = \frac{1}{2}\left(\frac{\Omega_B - \Omega_A}{\omega}\right)}$$

The above equation is similar to the one obtained by the previous method. In fact, for small damping β_A & β_B are very close to β_1 & β_2, respectively.

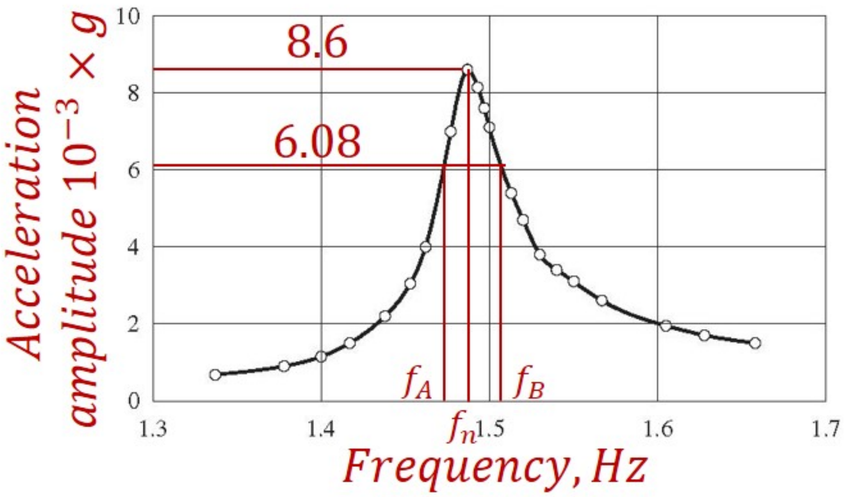
EXAMPLE:

The steady-state acceleration amplitude of a structure caused by an eccentric-mass vibration generator was measured for several excitation frequencies. These data are as follows:

Συχνότητα (Hz)	Επιτάχυνσις ($10^{-3}g$)	Συχνότητα (Hz)	Επιτάχυνσις ($10^{-3}g$)
1.337	0.68	1.500	7.10
1.378	0.90	1.513	5.40
1.400	1.15	1.520	4.70
1.417	1.50	1.530	3.80
1.438	2.20	1.540	3.40
1.453	3.05	1.550	3.10
1.462	4.00	1.567	2.60
1.477	7.00	1.605	1.95
1.487	8.60	1.628	1.70
1.493	8.15	1.658	1.50
1.497	7.60		

Determine the natural frequency and damping ratio of the structure.

SOLUTION:



Natural frequency

Phase resonance (*i.e.*, $\beta = 1$) occurs very near the peak (but not exactly at the peak). However, for small damping it is reasonable to say that the frequency response curve peaks at

$$\boxed{f_n = 1.487\ Hz}$$

where f_n is the natural frequency of the structure.

Damping ratio

The acceleration amplitude at the peak is $\rho_{peak} = 8.6 \times 10^{-3} g$.

We draw a horizontal line at the level $\rho_{peak}/\sqrt{2} = 6.08 \times 10^{-3} g$ to obtain the two frequencies f_A & f_B in Hz:

$$f_A = 1.473\ Hz \quad , \quad f_B = 1.507\ Hz$$

Then,

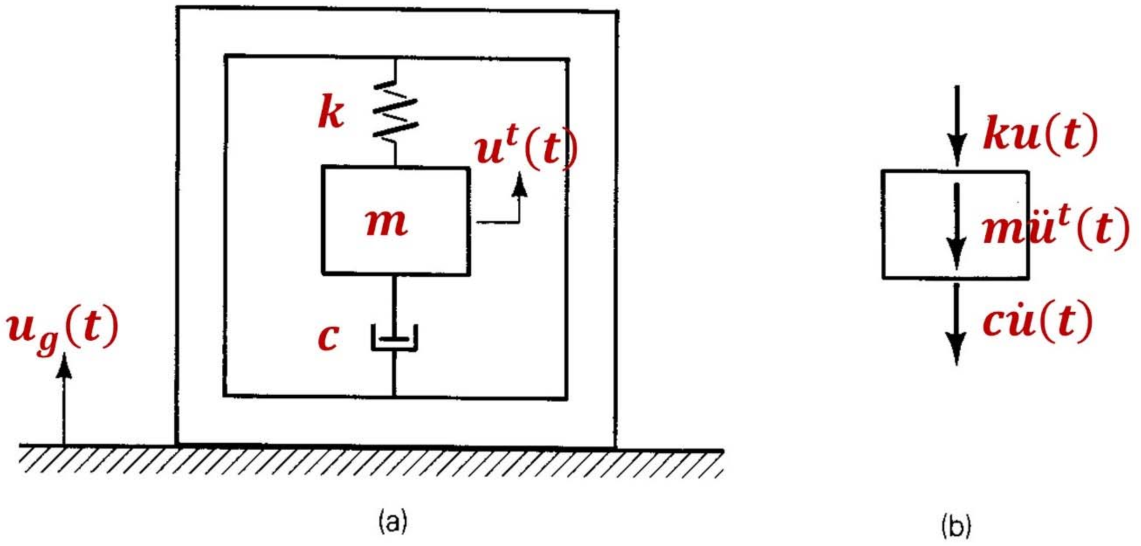
$$\xi = \frac{f_B - f_A}{2 f_n} = \frac{1.507 - 1.473}{2 \times 1.487} = 0.0114$$

Therefore

$$\boxed{\xi \quad = \quad 1.14\%}$$

TRANSMITTED MOTION DUE TO SUPPORT MOVEMENT

A system mounted on a moving support will have some of the support motion transmitted to it. Often the design of such a system requires that the **transmitted motion be minimized**.



Let: $u_g(t) = G\sin(\Omega t)$

Then: $u^t(t) = u(t) + u_g(t)$

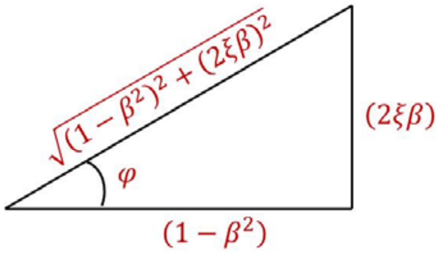
(The **superscript '*t*'** stands for '***total***'; the **subscript '*g*'** stands for '***ground***')

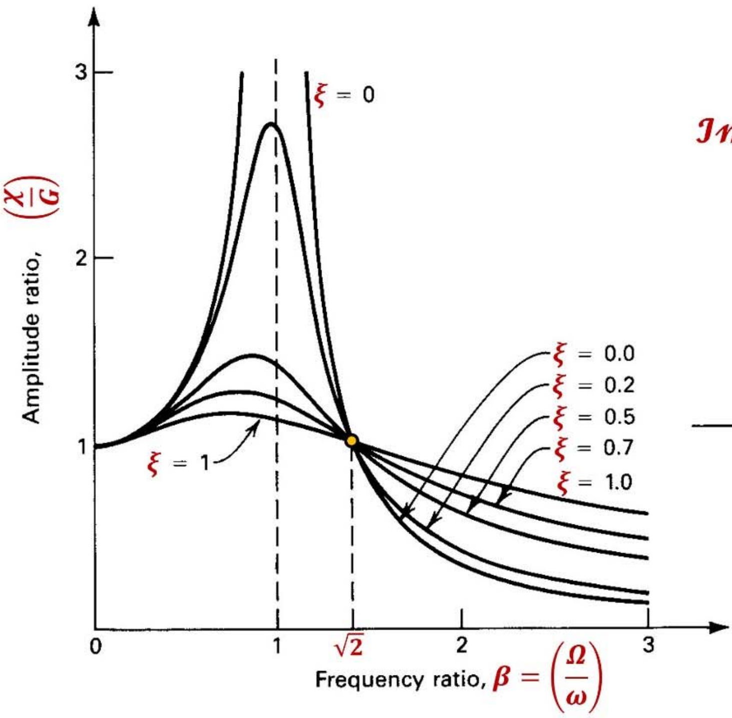
Equation of Motion: $m\ddot{u}^t + c\dot{u} + ku = 0$

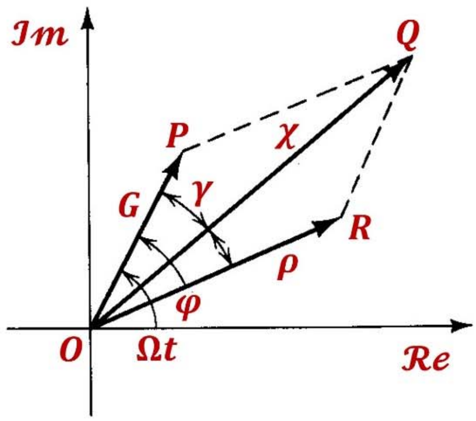
$$\Rightarrow \quad \begin{cases} m\ddot{u} + c\dot{u} + ku &= -m\ddot{u}_g \\ &= mG\Omega^2\sin(\Omega t) \end{cases}$$

Steady-state Response: $u_{ss}(t) = \rho\sin(\Omega t - \varphi)$

$$\text{where:} \begin{cases} \rho &= \left(\frac{mG\Omega^2}{k}\right)\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}} \\ &= G\frac{\beta^2}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}} \\ \tan\varphi &= \frac{2\xi\beta}{1-\beta^2} \end{cases}$$







Total displacement u^t:

$$\begin{aligned} u^t &= \underbrace{G\sin(\Omega t)}_{u_g(t)} + \underbrace{\frac{G\beta^2}{(1-\beta^2)^2+(2\xi\beta)^2}\{(1-\beta^2)\sin(\Omega t) - (2\xi\beta)\cos(\Omega t)\}}_{u(t)} \\ &= \frac{G}{(1-\beta^2)^2+(2\xi\beta)^2}\{[1-\beta^2+(2\xi\beta)^2]\sin(\Omega t) - (2\xi\beta^3)\cos(\Omega t)\} \\ &= \chi\sin(\Omega t - \gamma) \end{aligned}$$

where:

$$\chi = G\sqrt{\frac{1+(2\xi\beta)^2}{(1-\beta^2)^2+(2\xi\beta)^2}} \quad \& \quad \tan\gamma = \frac{(2\xi\beta^3)}{(1-\beta^2)^2+(2\xi\beta)^2}$$

The amplitude ratio (χ/G) vs. β is plotted above. The ratio (χ/G) is referred to as ***displacement transmissibility***.

The **rotating vector representation** can also be used quite effectively to obtain the **total displacement** u^t (see FIGURE above):

$$\begin{aligned} \chi^2 &= G^2 + \rho^2 - 2G\rho\cos(\pi - \varphi) \\ &= G^2 + \rho^2 + 2G\rho\cos\varphi \end{aligned}$$

where $$\cos\varphi = \frac{(1-\beta^2)}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}$$

Phase angle γ between G & χ is obtained by applying the **cosine identity** to the triangle $\triangle(OPQ)$: $\rho^2 = G^2 + \chi^2 - 2G\chi\cos\gamma$. This leads to the same value of γ as the one derived above.

TRANSMISSIBILITY AND VIBRATION ISOLATION

In practical design it is frequently required that the dynamic forces transmitted by a machine to its surroundings be minimized.

(*Force Isolation*)

Total transmitted force *F* (assuming **steady-state** response):

$$\begin{aligned} F &= f_S + f_D \\ &= k\rho\sin(\Omega t-\varphi) + c\Omega\rho\cos(\Omega t-\varphi) \\ &= F_0\sin(\Omega t-\varphi+\eta) \end{aligned}$$

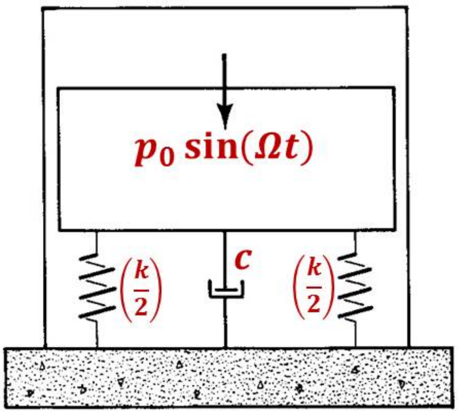
$$\text{where: } \begin{cases} F_0 = \sqrt{(k\rho)^2+(c\Omega\rho)^2} \\ \tan\eta = \dfrac{c\Omega}{k} \end{cases}$$

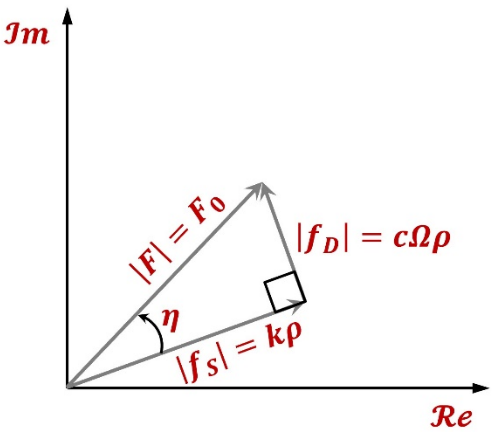
We know that: $\rho = \left(\frac{p_0}{k}\right)\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}$ & $\frac{c}{k} = \frac{2\xi}{\omega}$

It follows that:

$$F_0 = p_0\sqrt{\frac{1+(2\xi\beta)^2}{(1-\beta^2)^2+(2\xi\beta)^2}}$$
$$\tan\eta = 2\xi\beta$$

Note that the **force balance diagram**, along with the concept of **rotating vector representation**, can be utilized effectively to obtain **the transmitted force *F*** (see FIGURE below).

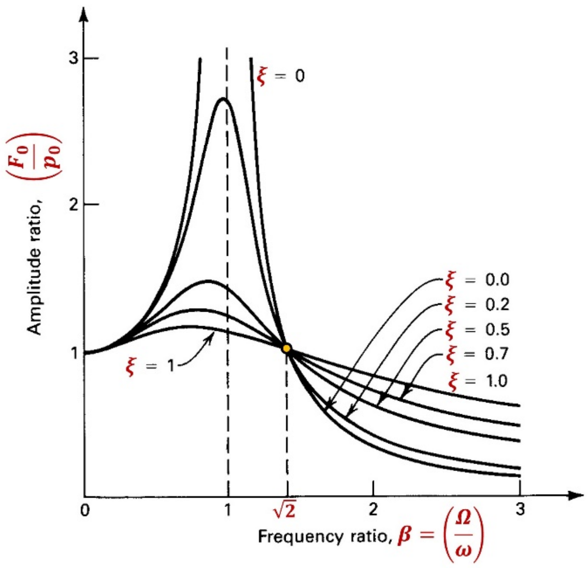




Transmission Ratio
or
Force Transmissibility $(TR)_f$

$$(TR)_f = \frac{F_0}{p_0} = \sqrt{\frac{1+(2\xi\beta)^2}{(1-\beta^2)^2+(2\xi\beta)^2}}$$

$(TR)_f$ *is a* ***measure*** *of the* ***force transmitted*** *to the foundation*

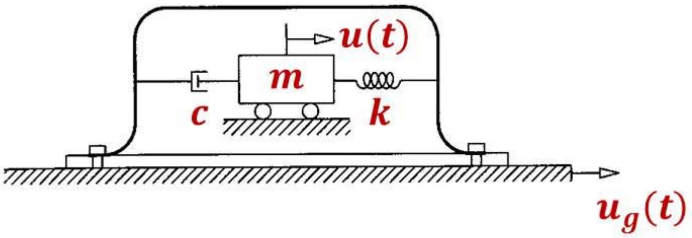


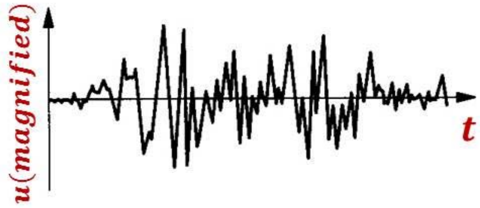
Note that the ratio (F_0/p_0) is the same as the ratio (χ/G) that we derived for the '***Transmitted motion due to support movement***'.

If the **transmitted force** $|F| = F_0$ is to be **smaller** than the **applied force** p_0, the natural frequency ω should be selected so that the frequency ratio $\beta = (\Omega/\omega) > \sqrt{2}$.

Also, **for** $\beta > \sqrt{2}$, the transmission ratio **decreases with damping**, so that theoretically, **zero damping will give the smallest transmitted force**. In practice, however, **some damping should always be provided to ensure that during startup as the machine passes through the resonant frequency, the response is kept within reasonable limits**.

MEASUREMENT OF ACCELERATION – ACCELEROMETER





Let: $\ddot{u}_g(t) = A\sin(\Omega t)$

Equation of Motion: $m\ddot{u} + c\dot{u} + ku = -mA\sin(\Omega t)$

Steady-state response: $u_{ss}(t) = \rho\sin(\Omega t - \varphi)$

$$\text{where: } \begin{cases} \rho = \left(\frac{mA}{k}\right)\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}} \\ \tan\varphi = \frac{2\xi\beta}{1-\beta^2} \end{cases}$$

If the instrument is **to be designed to measure an input acceleration** which may, in fact, have several harmonic components of different frequencies, **the measured displacement $u(t)$ should be proportional to the input for all values of the input frequency**.

$$\frac{\rho}{A} = \frac{1}{\omega^2}\cdot\frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}$$

For a satisfactory instrument design, $(\rho\omega^2/A)$ <u>should not vary with</u> β .

For $\xi = 0.7$, $(\rho\omega^2/A)$ stays **approximately constant** at a value of **1**, **provided that** $0 \le \beta \le 0.6$.

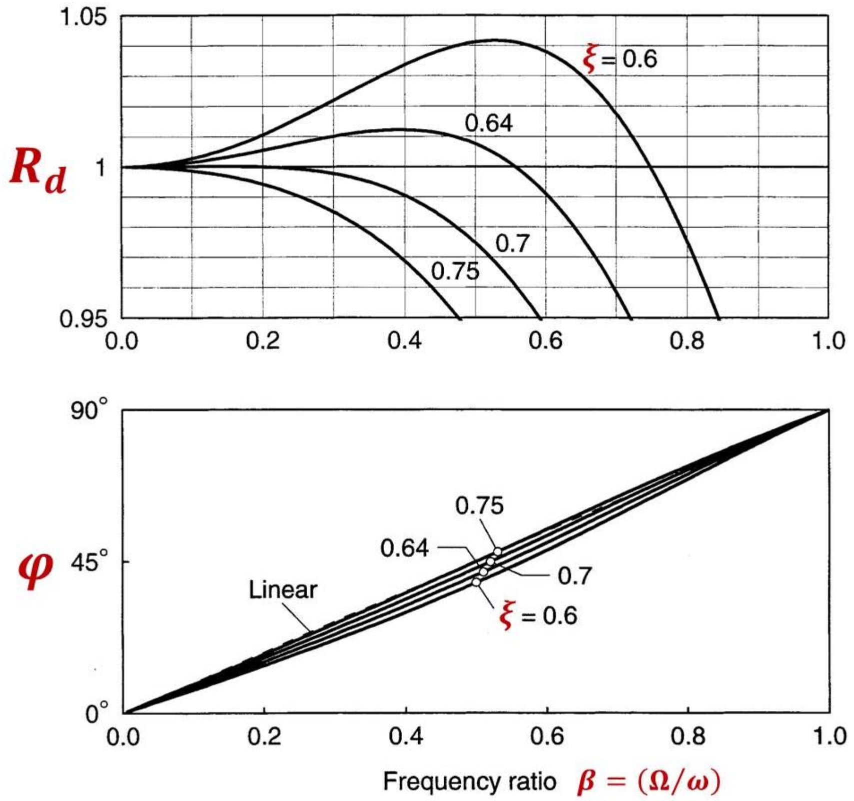
The **time shift for one harmonic** is:

$$t_s = \frac{\varphi}{\Omega} = \frac{1}{\Omega}\tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) = \frac{1}{\omega\beta}\tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

It is required that $t_s = \frac{\varphi}{\Omega}$ not vary with β.

For $\xi = 0.7$, (φ/Ω) is **practically constant**, *i.e.*, φ is a **linear function of Ω** and hence of β (see attached FIGURE).

Variation of R_d and φ with $\beta = (\Omega/\omega)$

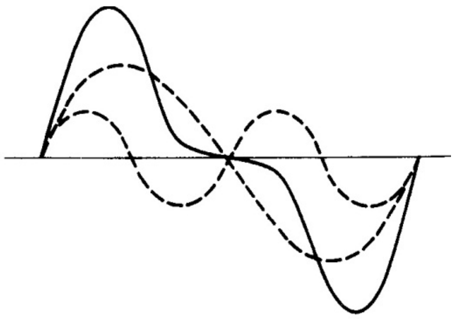


β	$(\rho\omega^2/A)$		(φ/β)
	$\xi = 0$	$\xi = 0.7$	$\xi = 0.7$
0	1.00	1.000	1.400
0.1	1.01	1.000	1.405
0.2	1.04	1.000	1.419
o.3	1.10	0.998	1.442
0.4	1.19	0.991	1.470
0.5	1.33	0.975	1.502
0.6	1.56	0.947	1.533

EXAMPLE:

Investigate the output of the accelerometer with damping $\boldsymbol{\xi = 0.70}$ when used to measure ground motion with the ground acceleration $\boldsymbol{\ddot{u}_g(t)}$ given by:

$$\ddot{u}_g(t) = Y_1 \sin(\Omega_1 t) + Y_2 \sin(\Omega_2 t)$$



For $\boldsymbol{\xi} = \mathbf{0.7}$, $\left(\boldsymbol{\varphi} \cong \left(\frac{\pi}{2}\right)\left(\frac{\Omega}{\omega}\right)\right)$ (where $\boldsymbol{\omega}$ is the **natural circular frequency of the accelerometer**), so that:

$$\boldsymbol{\varphi_1} \cong \left(\frac{\pi}{2}\right)\left(\frac{\boldsymbol{\Omega_1}}{\boldsymbol{\omega}}\right) \quad \& \quad \boldsymbol{\varphi_2} \cong \left(\frac{\pi}{2}\right)\left(\frac{\boldsymbol{\Omega_2}}{\boldsymbol{\omega}}\right)$$

[The natural circular frequency $\boldsymbol{\omega}$ of the instrument has been selected so that $\boldsymbol{\omega} \gg \boldsymbol{\Omega_1, \Omega_2}$.]

$$Y_1 \sin(\Omega_1 t) \quad \rightarrow \boxed{Accelerometer} \rightarrow \quad \left(\frac{Y_1}{\omega^2}\right) \sin\left[\Omega_1 t - \frac{\pi}{2}\left(\frac{\Omega_1}{\omega}\right)\right]$$

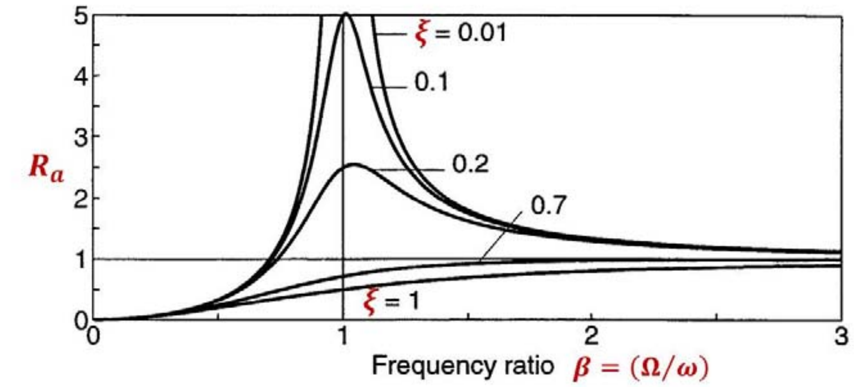
$$Y_2 \sin(\Omega_2 t) \quad \rightarrow \boxed{Accelerometer} \rightarrow \quad \left(\frac{Y_2}{\omega^2}\right) \sin\left[\Omega_2 t - \frac{\pi}{2}\left(\frac{\Omega_2}{\omega}\right)\right]$$

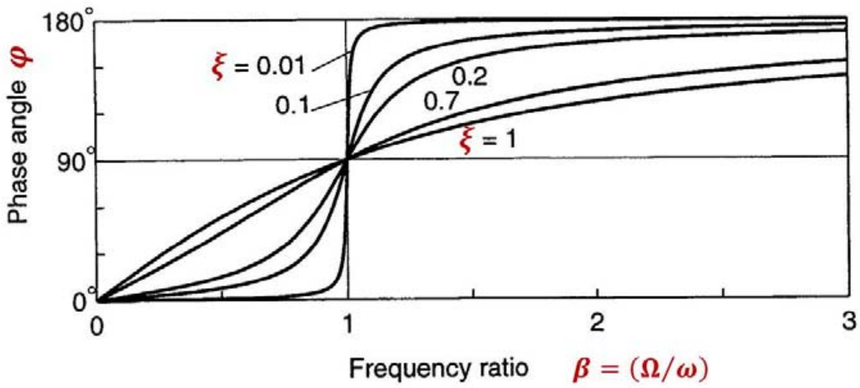
$$\left.\begin{matrix} Y_1 \sin(\Omega_1 t) \\ + \\ Y_2 \sin(\Omega_2 t) \end{matrix}\right\} \rightarrow \boxed{Accelerometer} \rightarrow \left\{\begin{matrix} \left(\frac{Y_1}{\omega^2}\right) \sin\left[\Omega_1 \left(t - \frac{\pi}{2\omega}\right)\right] \\ + \\ \left(\frac{Y_2}{\omega^2}\right) \sin\left[\Omega_2 \left(t - \frac{\pi}{2\omega}\right)\right] \end{matrix}\right.$$

Because the time functions in both terms are equal to $\left(\boldsymbol{t - \frac{\pi}{2\omega}}\right)$, **the <u>shift of both components along the time axis is equal</u>**. Thus, the instrument faithfully reproduces (within a multiplicative factor $\boldsymbol{\omega^2}$) the ground acceleration $\boldsymbol{\ddot{u}_g(t)}$.

MEASUREMENT OF DISPLACEMENT – DISPLACEMENT METER

The transducer must be designed so that the spring is so flexible or the mass so large, or both, that the mass stays still while the support beneath it moves. Such an instrument is unwieldy.





Support Displacement: $u_g(t) = u_{go}\sin(\Omega t) \quad \Longrightarrow \quad \ddot{u}_g(t) = -u_{go}\Omega^2\sin(\Omega t)$

Equation of Motion:

$$\begin{cases} m\ddot{u}^t + c\dot{u} + ku = 0 \quad \Longrightarrow \quad m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t) \\ \quad\quad\quad\quad\quad\quad\quad\quad\;\; \Longrightarrow \quad m\ddot{u} + c\dot{u} + ku = \underbrace{mu_{go}\Omega^2}_{(p_{eff})_o}\sin(\Omega t) \end{cases}$$

$$\left(p_{eff}\right)_o \sin(\Omega t) \;\rightarrow\; \boxed{\text{Displacement Meter}} \;\rightarrow\; \rho\sin(\Omega t - \varphi)$$

where: $$\rho = \frac{\left(p_{eff}\right)_o}{k}R_d = \frac{mu_{go}\Omega^2}{k}R_d = u_{go}\left(\frac{\Omega}{\omega}\right)^2 R_d = u_{go}R_a$$

For $\boldsymbol{\beta} = \left(\frac{\Omega}{\omega}\right) \gg \mathbf{1}$ (*i.e.*, **sufficiently large**), $\boldsymbol{R_a} \cong \mathbf{1}$ (*i.e.*, **independent of** β) and $\boldsymbol{\varphi} \cong \boldsymbol{\pi}$ (for sufficiently small damping). **This implies that although the measured motion is negative of the input motion, there is no shift along the time axis and the output motion, comprised of any number of harmonic components, will be reproduced correctly**.