VISCOUSLY DAMPED SDOF SYSTEM: RESPONSE TO HARMONIC EXCITATION

HARMONIC VIBRATION OF UNDAMPED SYSTEMS:

Equation of Motion: $\boxed{m\ddot{u} + ku = p_0 \sin(\Omega t)}$

Initial Conditions: $u_0 = u(t=0) \quad \& \quad \dot{u}_0 = \dot{u}(t=0)$

Solution: $u(t) = u_h(t) + u_p(t)$

For a particular solution, we try: $u_p(t) = G\sin(\Omega t)$

Substitution in the Equation of Motion:

$$G(k - m\Omega^2)\sin(\Omega t) = p_0 \sin(\Omega t)$$

$$\Longrightarrow \qquad G = \frac{p_0}{(k - m\Omega^2)} \qquad (\Omega \neq \omega)$$

Therefore:

$$u(t) = \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{u_h(t)} + \underbrace{\frac{p_0}{(k - m\Omega^2)}\sin(\Omega t)}_{u_p(t)}$$

NOTE: The **complementary solution** $u_h(t)$ represents a fraction of the **transient** part of the solution, and **for the slightest amount of damping resistance it dies out with time**. This will become evident when we consider damped harmonic response.

The unknown constants A & B are determined so that $u_h(t) + u_p(t)$ satisfies the initial conditions u_0 & $\dot{u}_0$. The reader is **warned against** simply assuming $A = u_0 \;\; \& \;\; B = \frac{\dot{u}_0}{\omega}$.

Introducing the initial conditions, we obtain:

$$u(t) \quad = \quad \underbrace{u_0 \cos(\omega t) + \left[\frac{\dot{u}_0}{\omega} - \left(\frac{p_0}{k}\right)\frac{\beta}{(1-\beta^2)}\right]\sin(\omega t)}_{TRANSIENT} + \underbrace{\left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)}\sin(\Omega t)}_{STEADY-STATE}$$

where: $$\boldsymbol{\beta = \frac{\Omega}{\omega}}$$ **frequency ratio**

Transient Vibration:

It depends on u_0 & $\dot{u}_0$

It exists even if $u_0 = \dot{u}_0 = 0$

Steady-State Vibration:

It may be expressed as follows,

$$u(t) = (u_{st})_0 \frac{1}{(1-\beta^2)} \sin(\Omega t)$$

where:

$$(u_{st})_0 = \frac{p_0}{k} \quad \text{static deformation}$$

Therefore:

$$\begin{aligned} u(t) \quad &= \quad (u_{st})_0 \frac{1}{|1-\beta^2|} \sin(\Omega t - \varphi) \\ &(u_{st})_0 \, R_d \sin(\Omega t - \varphi) \\ &\rho \sin(\Omega t - \varphi) \end{aligned}$$

where:

$$R_d \quad = \quad \text{deformation (or displacement) response factor}$$

$$\varphi \quad = \quad \begin{cases} 0 & \Omega < \omega \\ \pi & \omega < \Omega \end{cases} \quad \text{phase angle}$$

By following a procedure identical to that outlined above, it can be shown that when the excitation is

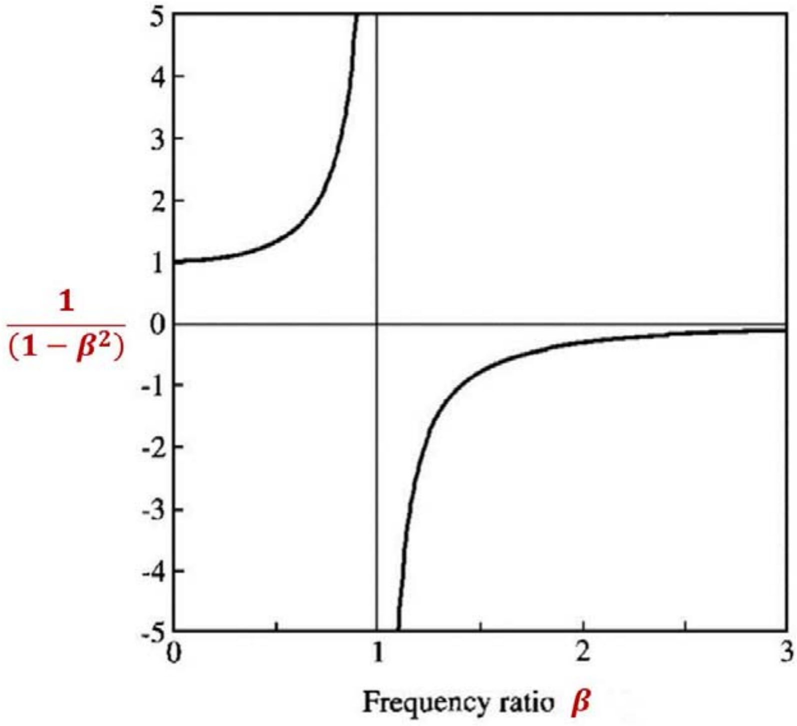
$$\boxed{p(t) = p_0 \cos(\Omega t)}$$

the **steady-state response** is given by

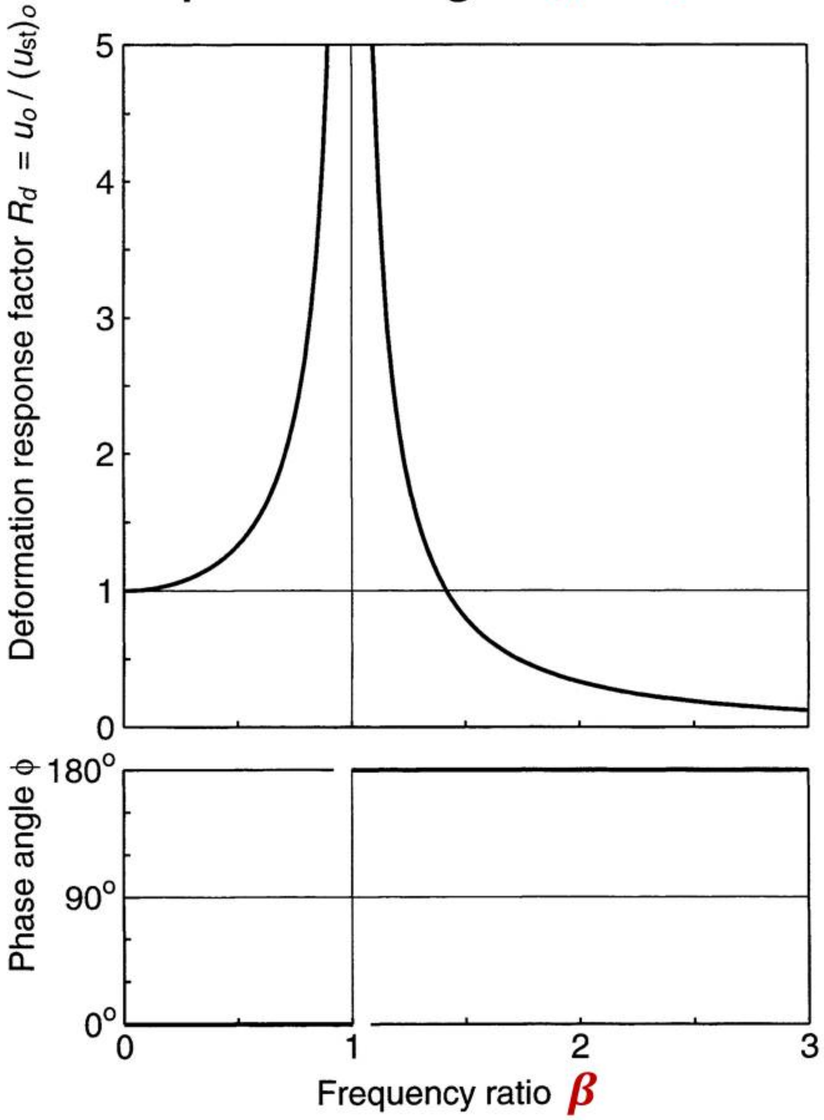
$$u_{ss}(t) = \left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)}\cos(\Omega t)$$

and the total response is given by

$$u(t) = \underbrace{\left[u_0 - \left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)}\right]\cos(\omega t) + \frac{\dot{u}_0}{\omega}\sin(\omega t)}_{TRANSIENT} + \underbrace{\left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)}\cos(\Omega t)}_{STEADY-STATE}$$



Deformation response factor and phase angle ($\xi = 0$)



RESONANT RESPONSE OF AN UNDAMPED SYSTEM:

We assume that the system starts from **rest**, *i.e.*, $u_0 = \dot{u}_0 = 0$ and that:

$$p(t) = p_0 \sin(\Omega t) = p_0 \sin(\beta \omega t)$$

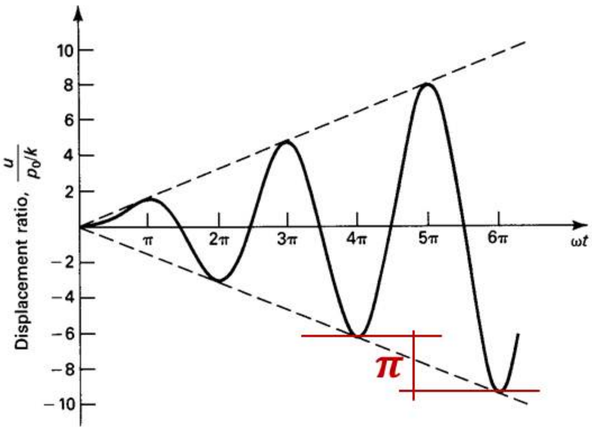
Then:

$$u(t) = \left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)}[\sin(\beta\omega t) - \beta \sin(\omega t)]$$

Resonance $\Leftrightarrow \beta = 1$:

$$\lim_{\beta\to 1} u(t) = \lim_{\beta\to 1}\left(\frac{p_0}{k}\right)\frac{\omega t\cos(\beta\omega t) - \sin(\omega t)}{-2\beta} \qquad L'Hospital's\ Rule$$

$$= \frac{1}{2}\left(\frac{p_0}{k}\right)[\sin(\omega t) - \omega t\cos(\omega t)]$$



The response is **periodic** with **period** $\left(\frac{2\pi}{\omega}\right)$. The **amplitude of the response continues to grow indefinitely**. It should be recognized that **after a short time the second term represents a good approximation to total response**, i.e., $u(t) \cong -\frac{p_0}{2k}\omega t\cos(\omega t)$.

A measure of the rate of growth may be obtained by taking the difference of amplitude at two successive peaks:

$$\dot{u} = \frac{p_0}{2k}\omega^2 t\sin(\omega t) \quad \Leftrightarrow \quad \omega t = n\pi \quad (n = 0,1,2,\cdots)$$

$\therefore$ Peaks occur at $t = \frac{n\pi}{\omega}$

$\therefore$ $u\left(\frac{2\pi}{\omega} + \frac{n\pi}{\omega}\right) - u\left(\frac{n\pi}{\omega}\right) = -\frac{p_0}{2k}2\pi\cos(n\pi) = \pm\frac{p_0}{k}\pi$

For general (*i.e.*, not necessarily zero) initial conditions, the response to $p(t) = p_0 \sin(\omega t)$ is given by

$$u(t) = u_0 \cos(\omega t) + \frac{\dot{u}_0}{\omega} \sin(\omega t) + \frac{p_0}{2k}[\sin(\omega t) - \omega t \cos(\omega t)]$$

Then:

$$\dot{u}(t) = -u_0 \omega \sin(\omega t) + \dot{u}_0 \cos(\omega t) + \frac{p_0 \omega^2}{2k} t \sin(\omega t)$$

For **large *t***:

$$u(t) \cong -\frac{p_0 \omega}{2k} t \cos(\omega t)$$

and

$$\dot{u}(t) \cong \frac{p_0 \omega^2}{2k} t \sin(\omega t)$$

Notice that $\dot{u}(t)$ is "***in phase***" with $(t) = p_0 \sin(\omega t)$ and, consequently, **power input is <u>always positive</u>**:

Power input	$p(t)\dot{u}(t)$	$\cong$	$\frac{p_0^2 \omega^2}{2k} t \sin^2(\omega t)$

RESONANT RESPONSE OF AN UNDAMPED SYSTEM (continued):

Alternative derivation of the response:

Excitation: $p(t) = p_0 \sin(\Omega t)$

Initial Conditions: $u(0) = \dot{u}(0) = 0$

Response: $u(t) = \left(\frac{p_0}{k}\right)\left(\frac{1}{1-\left(\frac{\Omega}{\omega}\right)^2}\right)\left[\sin(\Omega t) - \left(\frac{\Omega}{\omega}\right)\sin(\omega t)\right]$

When the forcing frequency $\boldsymbol{\Omega}$ is equal to or very close to the natural frequency $\boldsymbol{\omega}$ of the system, then we proceed as follows:

Let: $\omega - \Omega = 2\varepsilon$ (ε = **small quantity**)

We may re-write the response in the following form:

$$
\begin{aligned}
u(t) &= \left(\frac{p_0}{k}\right)\frac{\omega}{\omega^2-\Omega^2}\left\{\frac{\omega+\Omega}{2}[\sin(\Omega t)-\sin(\omega t)] + \frac{\omega-\Omega}{2}[\sin(\Omega t)+\sin(\omega t)]\right\} \\
&= \left(\frac{p_0}{k}\right)\frac{\omega}{\omega^2-\Omega^2}\left\{(\omega+\Omega)\cos\frac{(\Omega+\omega)t}{2}\sin\frac{(\Omega-\omega)t}{2} + (\omega-\Omega)\sin\frac{(\Omega+\omega)t}{2}\cos\frac{(\Omega-\omega)t}{2}\right\} \\
&= -\left(\frac{p_0}{k}\right)\frac{\omega}{2}\left\{\frac{\sin(\varepsilon t)}{\varepsilon}\cos[(\omega-\varepsilon)t] - \frac{\cos(\varepsilon t)}{\omega-\varepsilon}\sin[(\omega-\varepsilon)t]\right\}
\end{aligned}
$$

Evaluating the above equation in the limit, as $\varepsilon \to 0$, we have:

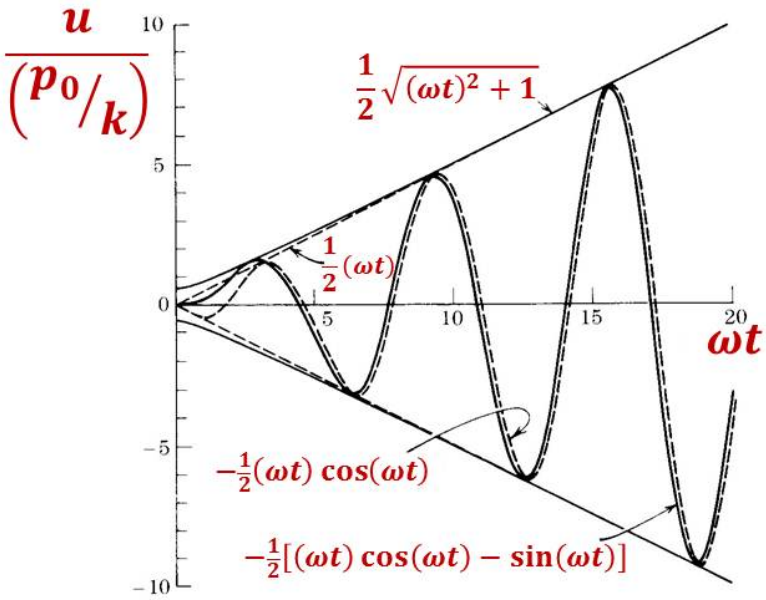
$$\lim_{\varepsilon \to 0} u(t) = -\frac{1}{2}\left(\frac{p_0}{k}\right)\{\omega t\cos(\omega t) - \sin(\omega t)\}$$

In **phase-angle form**, this expression becomes:

$$u(t) = -\left(\frac{p_0}{k}\right)\rho\cos(\omega t - \varphi)$$

$$\text{where} \quad \rho = \frac{1}{2}\sqrt{(\omega t)^2 + 1} \quad \& \quad \varphi = \tan^{-1}\left(\frac{-1}{\omega t}\right)$$

Therefore, in the limiting case **when $\Omega = \omega$, the amplitude of vibration ρ increases indefinitely with time** (see FIGURE).



The **solid** curve is a dimensionless plot of equation:

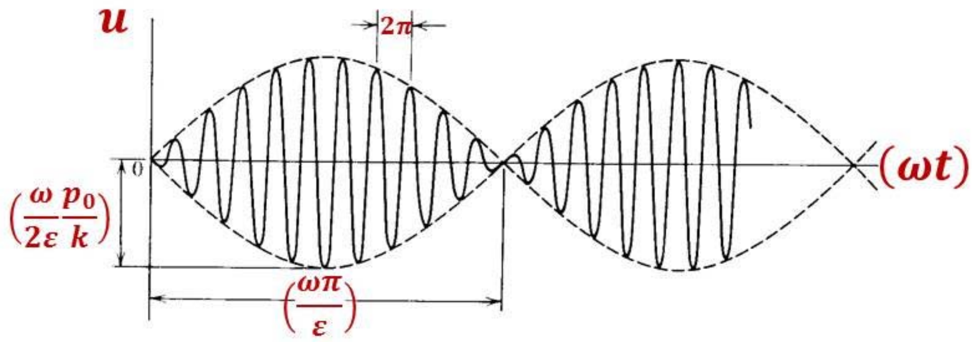
$$\lim_{\varepsilon \to 0} u(t) = -\frac{1}{2}\left(\frac{p_0}{k}\right)\{\omega t \cos(\omega t) - \sin(\omega t)\}$$

whereas the **dashed** curve is a similar plot of the **first term only**.

It can be seen that **after a short time the first term represents a good approximation to total response**, as follows:

$$\lim_{\varepsilon \to 0} u(t) \cong -\frac{\omega}{2}\left(\frac{p_0}{k}\right) t \cos(\omega t)$$

The curve in the FIGURE shows that the system theoretically attains an infinite amplitude of forced vibration at resonance in the absence of damping, but this amplitude requires infinite time to build up.



When the frequency, $\boldsymbol{\Omega}$**,** of the forcing function, is **close** to (but **not exactly equal** to) that of the vibrating system, a phenomenon known as ***beating*** may be observed.

Equation,

$$u(t) = -\left(\frac{p_0}{k}\right)\frac{\omega}{2}\left\{\frac{\sin(\varepsilon t)}{\varepsilon}\cos[(\omega-\varepsilon)t] - \frac{\cos(\varepsilon t)}{\omega-\varepsilon}\sin[(\omega-\varepsilon)t]\right\}$$

represents this situation, and **we may obtain a good approximation to the response by considering a simplified form of the first term**, as follows:

$$u(t) \cong -\left(\frac{p_0}{k}\right)\frac{\omega}{2}\frac{\sin(\varepsilon t)}{\varepsilon}\cos(\omega t)$$

Because the quantity $\boldsymbol{\varepsilon}$ is **small**, the function $\mathbf{sin}(\boldsymbol{\varepsilon t})$ **varies slowly**; and its period, equal to $(\mathbf{2}\boldsymbol{\pi}/\boldsymbol{\varepsilon})$, is **large**. Therefore, the above expression may be recognized as representing vibrations with period $(\mathbf{2}\boldsymbol{\pi}/\boldsymbol{\omega})$ and a **variable amplitude** equal to $(\boldsymbol{\omega}/\mathbf{2}\boldsymbol{\varepsilon})(\boldsymbol{p_0}/\boldsymbol{k})\,\mathbf{sin}(\boldsymbol{\varepsilon t})$. This kind of vibration builds up and diminishes in a regular pattern of ***beats***, as indicated in the FIGURE**. The period of beating, equal to** $(\boldsymbol{\pi}/\boldsymbol{\varepsilon})$, increases as $\boldsymbol{\Omega} \to \boldsymbol{\omega}$ (*i.e.*, as $\boldsymbol{\varepsilon} \to \mathbf{0}$). **At resonance, the period of beating becomes infinite**.

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NOTE: The phenomenon of beats is not uncommon in engineering structures. For example, see the following references:

Lin, B.C. and **A.S. Papageorgiou** (1989). "Demonstration of Torsional Coupling Caused by Closely Spaced Periods: 1984 Morgan Hill Earthquake, Response of the Santa Clara County Building", ***Earthquake Spectra***, Vol. **5**, No. **3**, pp. 539-556.

Papageorgiou, A.S. and B.C. Lin (1989). "Influence of Lateral-Load-Resisting System on the Earthquake Response of Structures - A System Identification Study", ***Earthquake Engineering and Structural Dynamics***, Vol. **18**, pp. 799-814.

HARMONIC VIBRATION WITH VISCOUS DAMPING:

Equation of Motion: $\boxed{m\ddot{u} + c\dot{u} + ku = p_0 \sin(\Omega t)}$

Initial Conditions: $u_0 = u(t=0) \quad \& \quad \dot{u}_0 = \dot{u}(t=0)$

Solution: $u(t) = u_h(t) + u_p(t)$

As a particular integral, we try a solution of the form:

$$u(t) = G_1 \cos(\Omega t) + G_2 \sin(\Omega t)$$

Introducing the above trial solution in the Equation of Motion, we obtain:

$$[-G_1\Omega^2 m + G_2\Omega c + G_1 k]\cos(\Omega t) + [-G_2\Omega^2 m - G_1\Omega c + G_2 k]\sin(\Omega t) = p_0 \sin(\Omega t)$$

$$\Rightarrow \begin{Bmatrix} -G_1\Omega^2 m + G_2\Omega c + G_1 k & = & 0 \\ -G_2\Omega^2 m - G_1\Omega c + G_2 k & = & p_0 \end{Bmatrix} \Rightarrow \begin{cases} (1-\beta^2)G_1 + 2\xi\beta G_2 & = & 0 \\ -2\xi\beta G_1 + (1-\beta^2)G_2 & = & \dfrac{p_0}{k} \end{cases}$$

$$\Rightarrow \begin{cases} G_1 & = & \left(\dfrac{p_0}{k}\right)\dfrac{-2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2} \\ G_2 & = & \left(\dfrac{p_0}{k}\right)\dfrac{(1-\beta^2)}{(1-\beta^2)^2 + (2\xi\beta)^2} \end{cases}$$

∴ ***Particular Solution***:

$$\boxed{u_p(t) = \left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\{(1-\beta^2)\sin(\Omega t) - 2\xi\beta\cos(\Omega t)\}}$$

Complete Solution:

$$u(t) = u_h(t) + u_p(t)$$

$$u(t) = \begin{cases} \underbrace{e^{-\xi\omega t}\{A\cos(\omega_d t) + B\sin(\omega_d t)\}}_{\textit{TRANSIENT}\ [u_t(t)]} \\ + \\ \underbrace{\left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\{(1-\beta^2)\sin(\Omega t) - 2\xi\beta\cos(\Omega t)\}}_{\textit{STEADY-STATE}\ [u_{ss}(t)]} \end{cases}$$

Introducing the **initial conditions** u_0 **&** $\dot{u}_0$, we obtain:

$$A = \left(\frac{p_0}{k}\right)\frac{2\xi\beta}{(1-\beta^2)^2 + (2\xi\beta)^2} + u_0$$

$$B = \left(\frac{p_0}{k}\right)\left(\frac{\omega}{\omega_d}\right)\left\{\frac{2\beta\xi^2 - \beta(1-\beta^2)}{(1-\beta^2)^2 + (2\xi\beta)^2}\right\} + \frac{\dot{u}_0 + \xi\omega u_0}{\omega_d}$$

Therefore, the **total response** is expressed as follows:

$$u(t) = \begin{cases} \underbrace{e^{-\xi\omega t}\left\{u_0\cos(\omega_d t) + \frac{\dot{u}_0 + \xi\omega u_0}{\omega_d}\sin(\omega_d t)\right\}}_{\textit{TRANSIENT}\ (\textit{related to IC})} \\ + \\ \underbrace{\left(\frac{p_0}{k}\right)\frac{e^{-\xi\omega t}}{(1-\beta^2)^2 + (2\xi\beta)^2}\left\{2\xi\beta\cos(\omega_d t) + \left(\frac{\omega}{\omega_d}\right)[2\beta\xi^2 - \beta(1-\beta^2)]\sin(\omega_d t)\right\}}_{\textit{TRANSIENT}\ (\textit{related to loading})} \\ + \\ \underbrace{\rho\sin(\Omega t - \varphi)}_{\textit{STEADY-STATE}} \end{cases}$$

Particular Solution/Integral:

$$u_p(t) = \left(\frac{p_0}{k}\right)\frac{1}{(1-\beta^2)^2 + (2\xi\beta)^2}\{(1-\beta^2)\sin(\Omega t) - 2\xi\beta\cos(\Omega t)\}$$

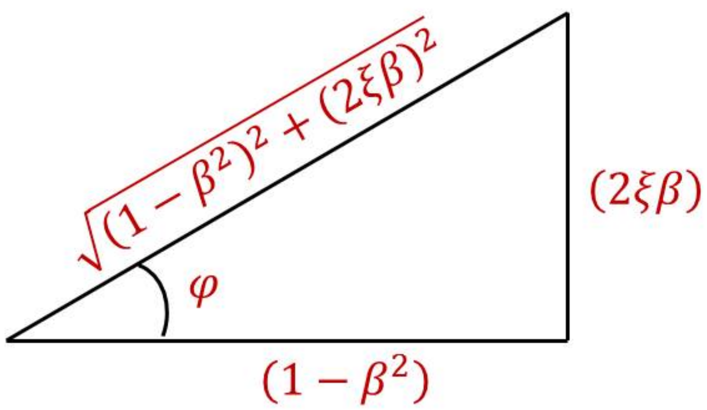
The **phase-angle form** of the particular solution:

$$u_p(t) = \rho\sin(\Omega t - \varphi)$$

where
$$\begin{cases} \rho = \left(\dfrac{p_0}{k}\right)\dfrac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \\ \tan\varphi = \dfrac{2\xi\beta}{1-\beta^2} \end{cases}$$

Displacement (or Deformation) Response Factor:

$$R_d \stackrel{\text{def}}{=} \frac{\rho}{(u_{st})_0} = \frac{\rho}{\left(\frac{p_0}{k}\right)} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$



$$\boldsymbol{Amplitude\ Resonance} \quad \Leftrightarrow \quad \max_{\beta} R_d(\beta,\xi)$$

$$\max_{\beta} R_d(\beta,\xi) \quad \Rightarrow \quad \frac{dR_d(\beta,\xi)}{d\beta} = -\frac{1}{2}\frac{-4\beta(1-\beta^2)+4\beta(2\xi^2)}{[(1-\beta^2)^2+(2\xi\beta)^2]^{3/2}} = 0$$

$$\Rightarrow \qquad \beta_{peak} = \sqrt{1-2\xi^2}$$

$$\therefore \qquad (R_d)_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

[**NOTE**: Observe that for $\xi > \frac{1}{\sqrt{2}}$ **no peak occurs** for R_d.]

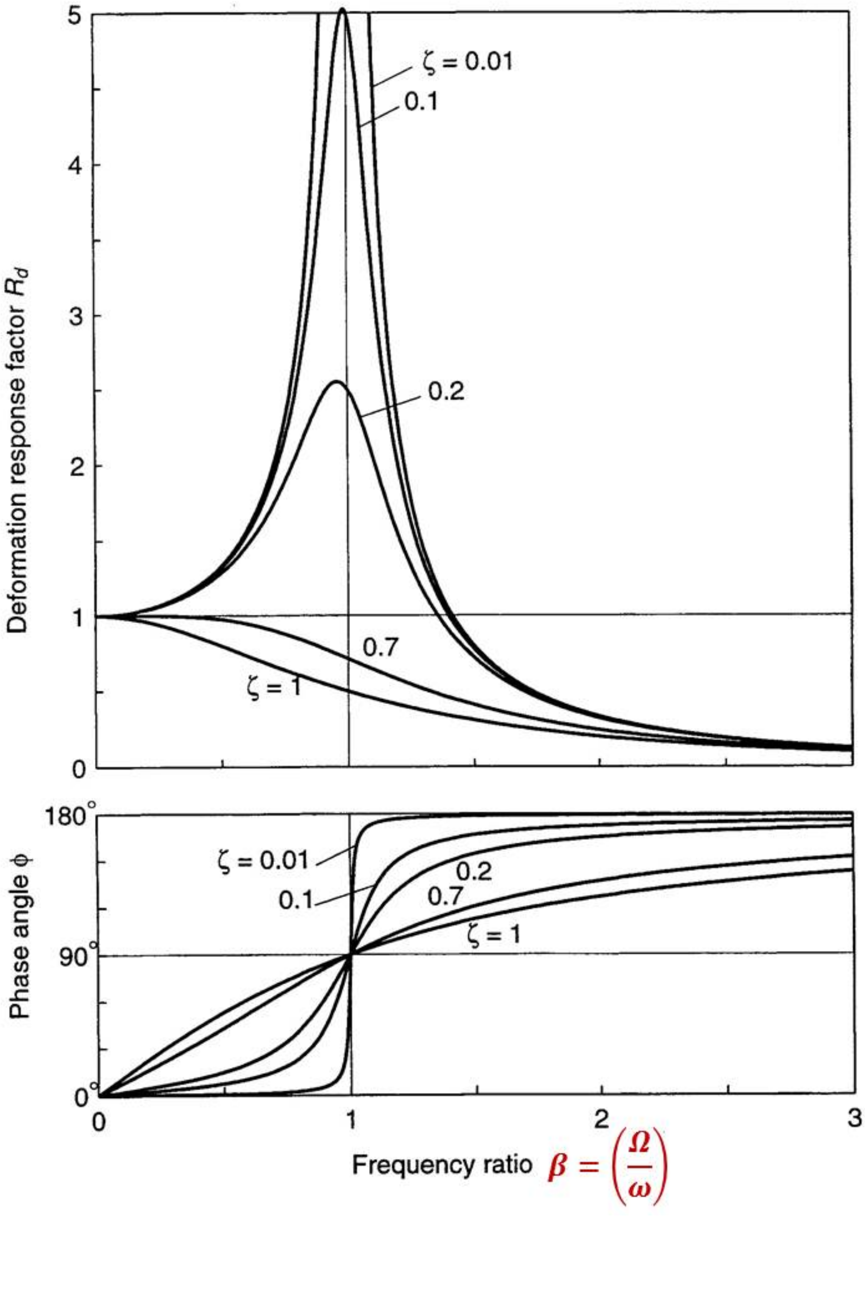
$$\boldsymbol{Phase\ Resonance} \quad \Leftrightarrow \quad \beta = 1 \quad i.e., \Omega = \omega$$

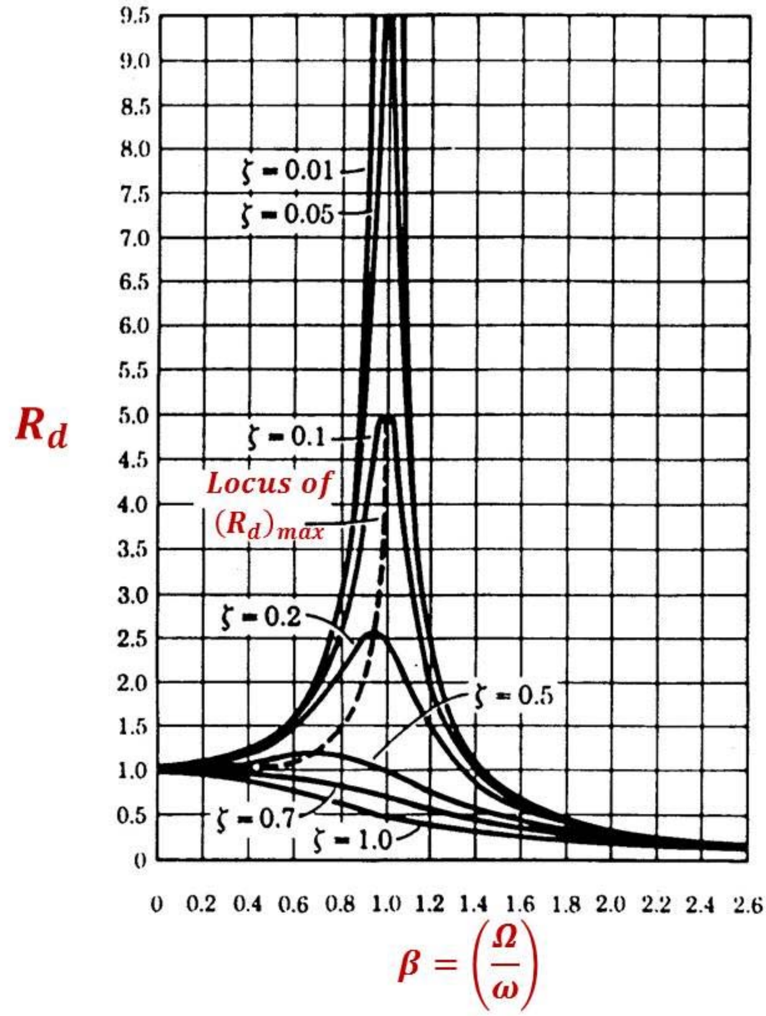
$$R_d(\beta = 1,\xi) = \frac{1}{2\xi}$$

Notice that $R_d(\beta = 1,\xi) < (R_d)_{max}$, i.e., **the maximum steady-state response** is achieved at **amplitude resonance** and not at phase resonance.

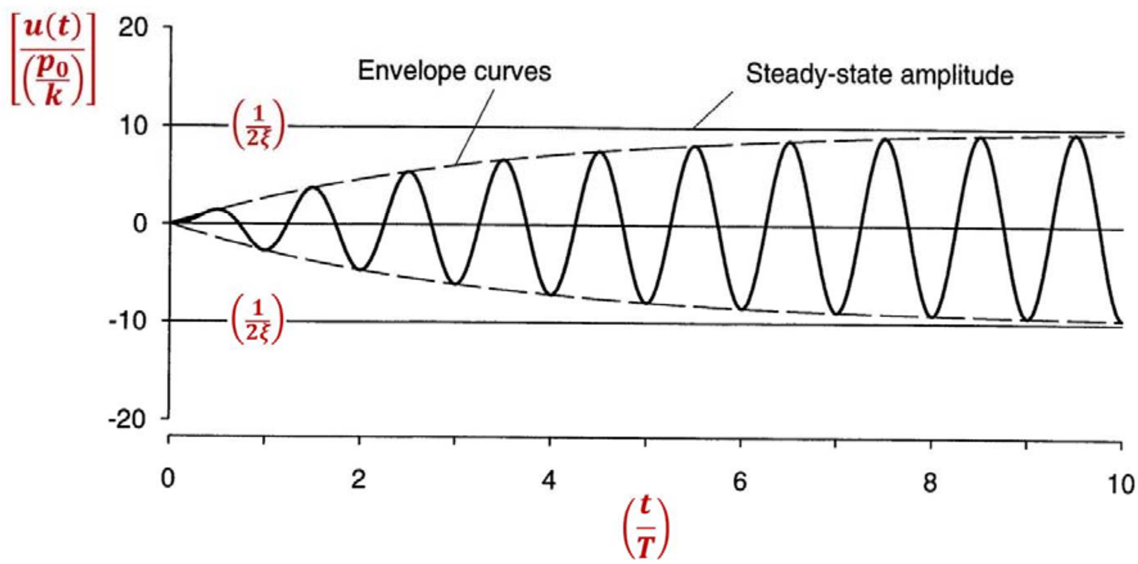
It must be recognized, however, **that the largest deformation peak may occur before the system has reached steady-state**.

Deformation response factor and phase angle

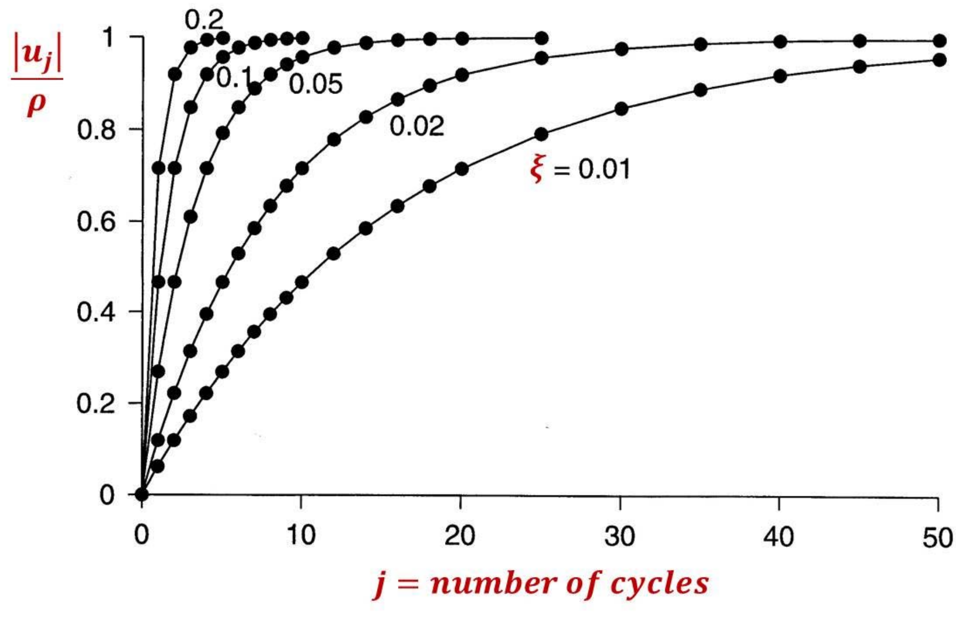



9.5
9.0
8.5
8.0
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
R_d
ζ = 0.01
ζ = 0.05
ζ = 0.1
Locus of
$(R_d)_{max}$
ζ = 0.2
ζ = 0.5
ζ = 0.7
ζ = 1.0
0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6
$\beta = \left(\frac{\Omega}{\omega}\right)$

Response to sinusoidal force with $\Omega = \omega \; (\xi = 5\%)$



Amplitude versus no. of cycles $(\Omega = \omega)$



We examine **the rate at which the steady-state is attained** in the case of **phase resonance**, *i.e.*, $\boldsymbol{\beta = 1 \Leftrightarrow \Omega = \omega}$.

Consider the viscously damped SDOF system starting from rest and subjected to $p(t) = p_0 \sin(\Omega t)$. The complete response is:

$$u(t) = \underbrace{\left(\frac{p_0}{k}\right)\left(\frac{1}{2\xi}\right)}_{\substack{\rho \\ = \text{steady-state} \\ \text{response} \\ \text{amplitude}}} \left\{e^{-\xi\omega t}\left[\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right] - \cos(\omega t)\right\}$$

For **lightly damped systems** (i.e., $\boldsymbol{\xi \ll 1}$) the term $\left[\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right]$ is **small** and $\boldsymbol{\omega_d = \omega\sqrt{1-\xi^2} \cong \omega}$; thus:

$$u(t) \cong \underbrace{\rho\left(e^{-\xi\omega t} - 1\right)}_{\substack{\text{envelope} \\ \text{function}}} \cos(\omega t) \quad \Longrightarrow \quad \boxed{\frac{|u_j|}{\rho} = 1 - e^{-2\pi\xi_j}}$$

where: $\boldsymbol{u_j}$ = **peak amplitude after $\boldsymbol{j}$ cycles of vibration**.

COMPLEX FREQUENCY RESPONSE FUNCTION

SDOF System with Viscous Damping:

Equation of Motion:

$$m\ddot{u} + c\dot{u} + ku = \underbrace{p(t)}_{1\cdot e^{i\Omega t}}$$

Steady-State Response:

$$u_{ss}(t) = H_u(\Omega)e^{i\Omega t}$$

- The **real part** of $H_u(\Omega)e^{i\Omega t}$ represents the response to the **real part** of $(1 \cdot e^{i\Omega t})$, i.e.,

$$\mathcal{Re}\{1 \cdot e^{i\Omega t}\} \rightarrow \boxed{SDOF} \rightarrow \mathcal{Re}\{H_u(\Omega)e^{i\Omega t}\}$$

- The **imaginary part** of $H_u(\Omega)e^{i\Omega t}$ represents the response to the **imaginary part** of $(1 \cdot e^{i\Omega t})$, i.e.,

$$\mathcal{Im}\{1 \cdot e^{i\Omega t}\} \rightarrow \boxed{SDOF} \rightarrow \mathcal{Im}\{H_u(\Omega)e^{i\Omega t}\}$$

Introducing $u_{ss}(t) = H_u(\Omega)e^{i\Omega t}$ in the Equation of Motion, we obtain:

$$H_u(\Omega)e^{i\Omega t}(-\Omega^2 m + i\Omega c + k) = e^{i\Omega t}$$

$$\Longrightarrow \boxed{H_u(\Omega) = \left(\frac{1}{k}\right) \cdot \frac{1}{(1-\beta^2) + i(2\xi\beta)}}$$

$H_u(\Omega)$ = **Complex Frequency-Response Function** (the subscript denotes that the function describes the **amplitude of the (harmonic) displacement**).

NOTE: Complex frequency-response functions can be similarly derived for other response quantities, quantities, *e.g.*, velocity $\dot{u}$, acceleration $\ddot{u}$, elastic restoring force $f_S = ku$, *etc.*

Notice that $H_u(\Omega)$, being a **complex number**, may be expressed as follows:

$$H_u(\Omega) = |H_u(\Omega)|e^{-i\varphi}$$

where:

$$|H_u(\Omega)| = \frac{\left(\frac{1}{k}\right)}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} = \left(\frac{1}{k}\right) R_d$$

$$\tan\varphi = \frac{-Im[H_u(\Omega)]}{Re[H_u(\Omega)]} = \frac{(2\xi\beta)}{(1-\beta^2)} \quad (0 \le \varphi \le \pi)$$

System Linearity (Superposition Principle):

$$1 \cdot e^{i\Omega t} \longrightarrow \boxed{SDOF} \longrightarrow H_u(\Omega) e^{i\Omega t}$$

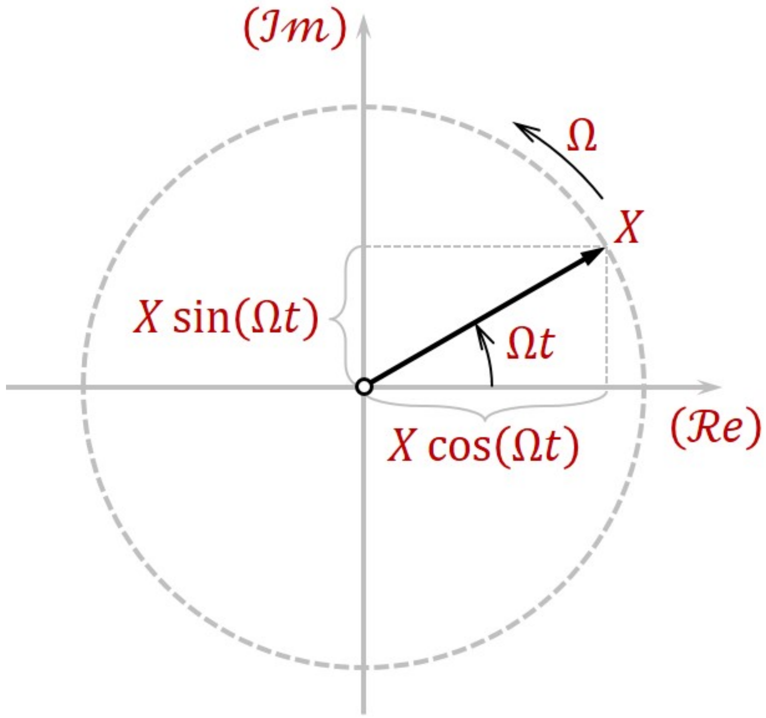
$$p_0 \cdot e^{i\Omega t} \longrightarrow \boxed{SDOF} \longrightarrow p_0 \cdot H_u(\Omega) e^{i\Omega t}$$

$$\underbrace{\sum_j \left((p_0)_j \cdot e^{i\Omega_j t}\right)}_{\textit{forcing function } \boldsymbol{p}(\boldsymbol{t})} \longrightarrow \boxed{SDOF} \longrightarrow \underbrace{\sum_j \left((p_0)_j \cdot H_u(\Omega_j) e^{i\Omega_j t}\right)}_{\textit{steady-state response } \boldsymbol{u_{ss}}(\boldsymbol{t})}$$

Therefore, the steady-state response $u_{ss}(t)$ to $p(t) = \sum_j (p_0)_j e^{i\Omega_j t}$ may be expressed as follows:

$$\begin{aligned} u_{ss}(t) &= \sum_j (p_0)_j H_u(\Omega_j) e^{i\Omega_j t} \\ &= \sum_j (p_0)_j \left|H_u(\Omega_j)\right| e^{-i\varphi_j} e^{i\Omega_j t} \\ &= \sum_j (p_0)_j \left(\frac{1}{k}\right) (R_d)_j e^{(i\Omega_j t - i\varphi_j)} \\ &= \sum_j (\rho)_j e^{(i\Omega_j t - i\varphi_j)} \end{aligned}$$

VECTOR AND COMPLEX NUMBER REPRESENTATION OF HARMONIC MOTIONS



It is convenient to **represent** a harmonic motion by means of a **rotating vector** of constant magnitude ***X*** at a constant **angular velocity** $\boldsymbol{\Omega}$.

A mathematically convenient way to represent such a rotating vector is using a complex number representation in the **complex** (***Argand***) **plane** as follows:

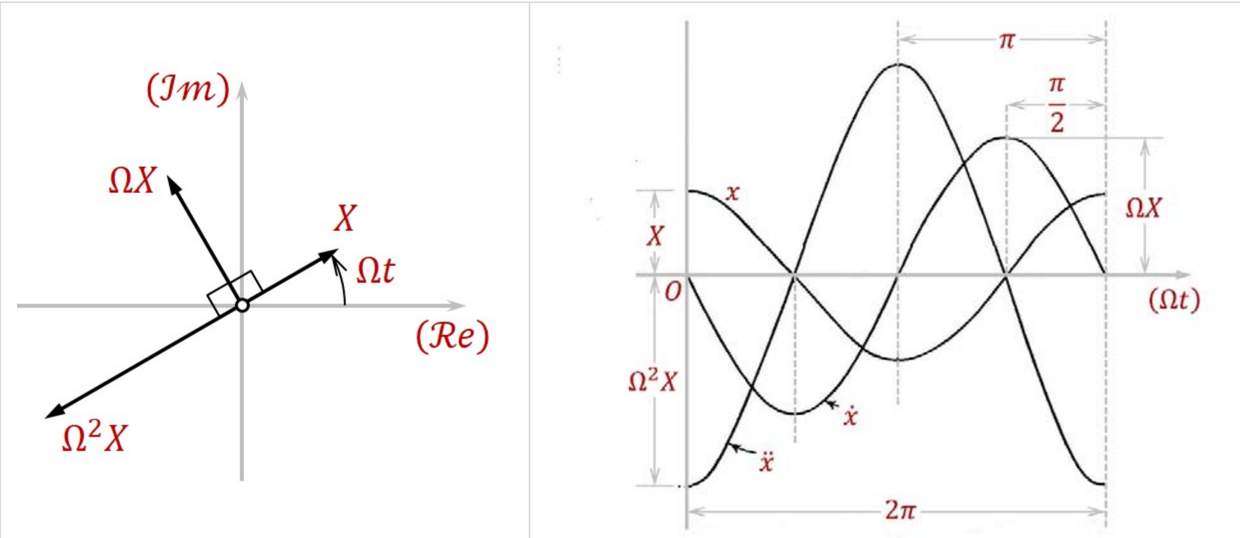
$$\left.\begin{aligned} z &= X\cos(\Omega t) + iX\sin(\Omega t) \\ &= Xe^{i\Omega t} \end{aligned}\right\} \quad \textbf{\textit{phasor}}\ (= \textbf{\textit{phase vector}})$$

The **projection of the rotating vector** on the ***real*** $-$ (*horizontal*$-$) axis is:

$$\mathcal{Re}\left[Xe^{i\Omega t}\right] = X\cos(\Omega t)$$

The **projection of the rotating vector** on the ***imaginary*** $-$ (*vertical*$-$) axis is:

$$\mathcal{Im}\left[Xe^{i\Omega t}\right] = X\sin(\Omega t)$$



If a harmonic displacement is $x(t) = X\cos(\Omega t)$, then:

Displacement: $x(t) = \mathcal{R}e\left[Xe^{i\Omega t}\right] = X\cos(\Omega t)$

Velocity: $\dot{x}(t) = \mathcal{R}e\left[i\Omega Xe^{i\Omega t}\right] = \mathcal{R}e\left[\Omega Xe^{i(\Omega t+\frac{\pi}{2})}\right] = \Omega X\cos\left(\Omega t + \frac{\pi}{2}\right)$

Acceleration: $\ddot{x}(t) = \mathcal{R}e\left[(i\Omega)^2 Xe^{i\Omega t}\right] = \mathcal{R}e\left[\Omega^2 Xe^{i(\Omega t+\pi)}\right] = \Omega^2 X\cos(\Omega t + \pi)$

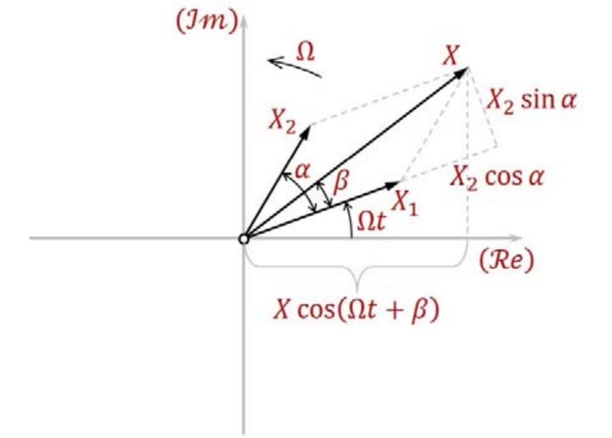
NOTE: Recall that, $i = e^{i\left(\frac{\pi}{2}\right)}$, and $i^2 = i\cdot i = e^{i\left(\frac{\pi}{2}\right)}\cdot e^{i\left(\frac{\pi}{2}\right)} = e^{i\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = e^{i\pi}$

The representation of displacement, velocity and acceleration by rotating vectors is illustrated in the FIGURE above.

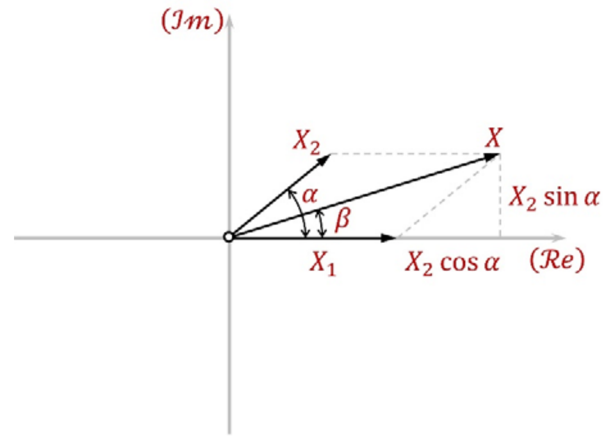
Since the given displacement $x(t) = X\cos(\Omega t)$ is a cosine function, (i.e., the projections of the corresponding rotating vector along the real axis), the velocity and acceleration must be also along the real axis. Hence the real parts of the respective rotating vectors give the physical quantities at a given time t.

If the given harmonic displacement is a sine function (*i.e.*, $x(t) = X\sin(\Omega t)$), then the physical quantities at a given time t are the imaginary parts of the respective rotating vectors (*i.e.*, projection of vectors on the vertical axis).

Addition Of Two Harmonic Motions



Vector addition



Vector addition with $\Omega t = 0$

Harmonic motions can be added algebraically (and graphically/geometrically) by means of vector addition.
As an example, let us consider **two harmonic motions having the <u>same circular frequency</u> Ω**:

$$x_1(t) = X_1 \cos(\Omega t) \qquad x_2(t) = X_2 \cos(\Omega t + \alpha)$$

The complex number/vector representation is:

$$z_1 = X_1 e^{i\Omega t} \qquad z_2 = X_2 e^{i(\Omega t + \alpha)}$$

The sum of the above two harmonic motions/functions may be expressed as follows:

$$\begin{aligned} z &= z_1 + z_2 \\ &= X_1 e^{i\Omega t} + X_2 e^{i(\Omega t+\alpha)} \\ &= \left(X_1 + X_2 e^{i\alpha}\right) e^{i\Omega t} \\ &= (X_1 + X_2 \cos\alpha + iX_2 \sin\alpha) e^{i\Omega t} \\ &= X e^{i\beta} e^{i\Omega t} \\ &= X e^{i(\Omega t + \beta)} \end{aligned}$$

where: $\begin{cases} X = \sqrt{(X_1 + X_2 \cos\alpha)^2 + (X_2 \sin\alpha)^2} \\ \beta = \tan^{-1} \dfrac{X_2 \sin\alpha}{X_1 + X_2 \cos\alpha} \end{cases}$

Since the given harmonic motions, $x_1(t)$ & $x_2(t)$, are along the real axis, their sum is:

$$x(t) = \mathcal{R}e[z] = \mathcal{R}e\left[X e^{i(\Omega t + \beta)}\right] = X \cos(\Omega t + \beta)$$

Evidently, **the sum of two harmonic motions of the <u>same</u> frequency, but with different phase angles, is itself a harmonic motion of the same frequency**.

However, **the sum of two harmonic motions of <u>different</u> frequencies is <u>not harmonic</u> <u>or</u>, in general, <u>periodic</u>**.

FORCE-BALANCE DIAGRAM OF STEADY-STATE RESPONSE

$$p(t) = p_0 e^{i\Omega t} \longrightarrow \boxed{SDOF} \longrightarrow u_{ss}(t) = \rho e^{i(\Omega t - \varphi)}$$

where:

$$\rho = \left(\frac{p_0}{k}\right) R_d$$

$$\varphi = \tan^{-1}\frac{(2\xi\beta)}{(1-\beta^2)} \quad (0 \le \varphi \le \pi)$$

Spring Force: $f_S = k u_{ss}(t) = k\rho e^{i(\Omega t - \varphi)}$

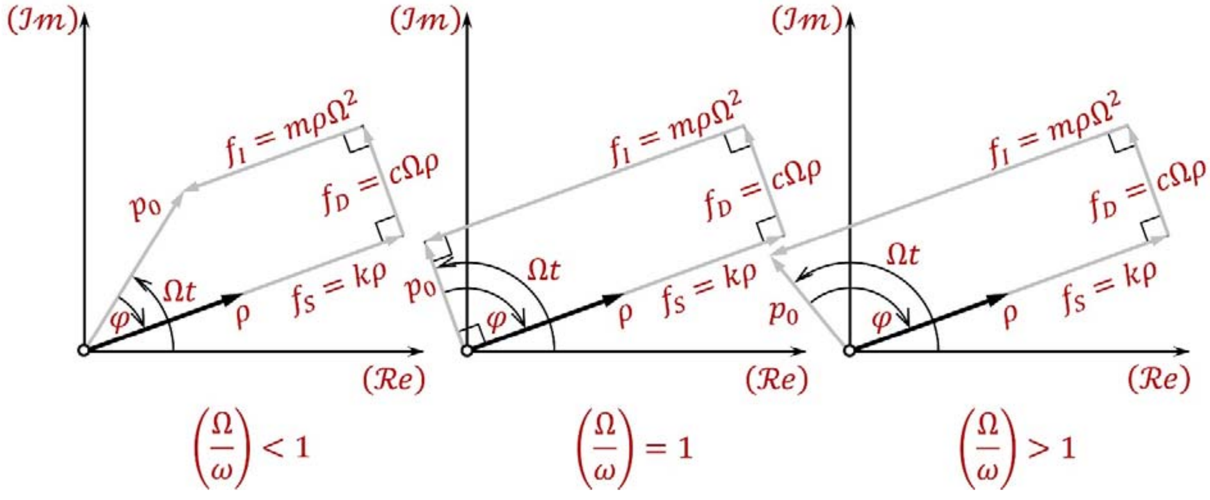
Damping Force: $f_D = c\dot{u}_{ss}(t) = c\rho\Omega e^{i\left(\Omega t - \varphi + \frac{\pi}{2}\right)}$

Inertia Force: $f_I = m\ddot{u}_{ss}(t) = m\rho\Omega^2 e^{i(\Omega t - \varphi + \pi)}$

To compare the magnitudes of $\boldsymbol{f_S, f_D, f_I}$ for the various ranges of $\boldsymbol{\beta} = \left(\frac{\Omega}{\omega}\right)$, divide the magnitudes of the vectors by $\boldsymbol{k}$:

i.e., $\frac{|f_S|}{k} = \rho$, $\frac{|f_D|}{k} = 2\xi\beta\rho$, $\frac{|f_I|}{k} = \beta^2\rho$, $\frac{|p(t)|}{k} = \frac{p_0}{k}$

and recall that: $f_I + f_D + f_S - p = 0$, (***dynamic equilibrium***)



RESPONSE TO PERIODIC LOADINGS

Any **periodic loading of period** T_0 [$i.e.$, $p(t + nT_0) = p(t)$] can be written as a sum of exponentials:

$$p(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\Omega_0 t}$$

where:

$$\Omega_0 = \frac{2\pi}{T_0}$$

$$C_n = \frac{1}{T_0}\int_0^{T_0} p(t) e^{-in\Omega_0 t}\, dt$$

For **real periodic functions** $\boldsymbol{p(t)}$: $\boldsymbol{C_{-n} = C_n^*}$

(the "*****" indicates **complex conjugate**)

We recall that:

$$p(t) = 1 \cdot e^{i\Omega t} \longrightarrow \boxed{SDOF} \longrightarrow u_{ss}(t) = H_u(\Omega) e^{i\Omega t}$$

where:

$$H_u(\Omega) = \left(\frac{1}{k}\right) \cdot \frac{1}{(1-\beta^2) + i(2\xi\beta)}$$

complex frequency response

Therefore, the **steady-state response** to a **periodic loading** $\boldsymbol{p(t)}$ is given **by superposition**:

$$\underbrace{\sum_{n=-\infty}^{+\infty} C_n e^{in\Omega_0 t}}_{\boldsymbol{p(t)}} \longrightarrow \boxed{SDOF} \longrightarrow \underbrace{\sum_{n=-\infty}^{+\infty} C_n H_u(n\Omega_0) e^{in\Omega_0 t}}_{\boldsymbol{u_{ss}(t)}}$$