Exercise 1 - Random Variable, Independence, Expected Value

Thursday, 7 December 2023 01:38

Suppose that two teams, A and B, play a series of matches that ends when one of the two teams has won two matches. The probability that team A will win a game is equal to p, there are no ties, and the outcome of each game is independent of the outcomes of the other games. Find the expected value of the number of games played.

Let A be the event A winning $\rightarrow p$

Similarly, B " B winning $\rightarrow q = 1-p$

Possibilities AA, ABA, BAA, BB, BAB, ABB

$p^2 \quad p^2q \quad p^2q \quad q^2 \quad pq^2 \quad pq^2$

Let X r.v. counting the number of matches

$X = x,\ x = 2,3 \Rightarrow P(X=2) = p^2+q^2$

$P(X=3) = 2p^2q + 2pq^2$

$EX = 2\cdot(p^2+q^2) + 3(2p^2q + 2pq^2) = 2p^2 + 2q^2 + 6pq(p+q) = \ldots$

$EX = \sum_{x=-\infty}^{\infty} x\, P(X=x) =$

The random variable's p.d.f is the following:

$$f(x) = \begin{cases} cx^2, 0 \le x \le 1 \\ 0, x < 0 \end{cases}$$

Calculate: (a) the constant c, (b) EX^2, (c) $E\left[(X-1)^2\right]$ (d) Variance of X

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 cx^2\,dx = 1 \Rightarrow c\left[\frac{x^3}{3}\right]_0^1 = 1 \Rightarrow$

$\Rightarrow c\left(\frac{1^3}{3} - \frac{0^3}{3}\right) = 1 \Rightarrow \frac{c}{3} = 1 \Rightarrow \boxed{c=3}$

β) $EX^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot 3x^2\,dx = 3\int_0^1 x^4\,dx = 3\left[\frac{x^5}{5}\right]_0^1 = 3\left(\frac{1^5}{5} - \frac{0^5}{5}\right) \Rightarrow$

$\Rightarrow \boxed{EX^2 = \frac{3}{5}}$

γ) $E\left[(X-1)^2\right] = E\left[X^2 - 2X + 1\right] \Rightarrow$ Linearity

$\Rightarrow E\left[(X-1)^2\right] = EX^2 - 2EX + 1 \quad (1)$

$EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x \cdot 3x^2\,dx = 3\int_0^1 x^3\,dx = 3\left[\frac{x^4}{4}\right]_0^1 =$

$= 3\left(\frac{1^4}{4} - \frac{0^4}{4}\right) \Rightarrow \boxed{EX = \frac{3}{4}}$

$(1) \Rightarrow E\left[(X-1)^2\right] = \frac{3}{5} - 2\frac{3}{4} + 1 = \frac{3}{5} - \frac{3}{2} + 1 = \frac{6 - 15 + 10}{10} \Rightarrow$

$$\boxed{E\left[(X-1)^2\right] = \frac{1}{16}}$$

δ) $\operatorname{Var}X = EX^2 - (EX)^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$

Box A contains 3 red and 2 blue balls, while box B contains 2 red and 8 blue balls. We toss a fair coin and if the result is "Heads," we take a random ball from the first box, otherwise we take a random ball from the second box.
a) What is the probability that the ball we took out is blue?
b) If the ball we took out is red, what is the probability that we took it out of box A?

a) Let A be the event choosing box A
Let $\bar{A}$ " " " " B

$P(A) = \frac{1}{2} \Rightarrow P(\bar{A}) = \frac{1}{2}$

Let M be the event selecting a blue ball.
$\bar{M}$ " " " red "

$P(M) = ?$

$P(M|A) = \frac{2}{5}$ $\qquad P(M|\bar{A}) = \frac{8}{10}$

Total Probability Theorem

$A, \bar{A}$ partition $\Rightarrow P(M) = P(M|A)\cdot P(A) + P(M|\bar{A})\cdot P(\bar{A}) =$

$= \frac{2}{5}\cdot\frac{1}{\cancel{2}} + \frac{\cancel{8}}{\cancel{10}}\cdot\frac{1}{\cancel{2}} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$

B) $P(A|\bar{M}) = ?$

Θ. Bayes: $P(A|\bar{M}) = \frac{\overbrace{P(\bar{M}|A)}^{1-P(M|A)}\cdot P(A)}{P(\bar{M})} = \frac{(1-P(M|A))\cdot P(A)}{1-P(M)} =$

$= \frac{\frac{3}{\cancel{5}}\cdot\frac{1}{2}}{\frac{2}{\cancel{5}}} \Rightarrow \boxed{P(A|\bar{M}) = \frac{3}{4}}$

Exercise 4 - Binomial + Gauss

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The probability of a player winning a game is 0.6. If he plays 100 independent games, find a) exactly and b) approximately, the probability of winning at most 70 and at least 55 games.

Let X r.v. counting the number of games played

$X \sim \text{Binomial}(n=100, p=0.6)$

$P(X=x) = \binom{100}{x} 0.6^{x} \cdot 0.4^{100-x}$

a) $P(55 \leq X \leq 70) = \sum_{x=55}^{70} \binom{100}{x} 0.6^{x} 0.4^{100-x}$

b) Approximation using normal

$\mu = n \cdot p = 100 \cdot 0.6 = 60$

$\sigma^2 = n \cdot p \cdot (1-p) = 100 \cdot 0.6 \cdot 0.4 = 24 \Rightarrow \sigma = 2\sqrt{6}$

Continuity correction

$P(55 \leq X \leq 70) \simeq P(55-0.5 < X < 70+0.5) = P(54.5 < X < 70.5) =$

$= P\left(\frac{54.5-60}{2\sqrt{6}} < \frac{X-60}{2\sqrt{6}} < \frac{70.5-60}{2\sqrt{6}}\right) =$

$= P(-1.123 < Z < 2.14) = P(Z < 2.14) - P(Z < -1.123) =$

$= \Phi(2.14) - \Phi(-1.123) =$

$= \Phi(2.14) - (1 - \Phi(1.123)) =$

$= \Phi(2.14) + \Phi(1.123) - 1$

The probability of a side effect to a drug is 0.001. Calculate the probability that, among 2,000 patients taking the drug, more than 2 patients will experience the side effect a) exactly b) approximately.

$p = 0.001$
$n = 2000$

a) Let X r.v. counting the number of people experiencing the side effect out of 2000

$X \sim Binomial(n=2000, p=0.001)$

$$P(X>2) = 1 - P(X \leq 2) = 1 - \sum_{x=0}^{2} P(X=x) = 1 - \left(P(X=0) + P(X=1) + P(X=2)\right) =$$

$$= 1 - \left[\binom{2000}{0} 0.001^0 \cdot 0.999^{2000} + \binom{2000}{1} 0.001^1 \cdot 0.999^{1999} + \binom{2000}{2} 0.001^2 \cdot 0.999^{1998}\right]$$

B) Approximation using Poisson

$\lambda = n \cdot p = 2000 \cdot 0.001 = 2$

$$P(X>2) = 1 - P(X \leq 2) = 1 - \sum_{x=0}^{2} P(X=x) = 1 - \left(P(X=0) + P(X=1) + P(X=2)\right) =$$

$$= 1 - \left(e^{-2} \cdot \frac{2^0}{0!} + e^{-2} \cdot \frac{2^1}{1!} + e^{-2} \cdot \frac{2^2}{2!}\right) = 1 - e^{-2}(1 + 2 + 2) =$$

$$= 1 - e^{-2} \cdot 5$$

For a random variable X the expected value is 3 and the second moment at the origin is 13. Calculate an upper bound for the probability $\Pr\{-2 < X < 8\}$.

$EX = 3$

Chebyshev: $\Pr\{|X - EX| \geq t\} \leq \frac{VarX}{t^2}$

$M''(0) = 13 \Rightarrow EX^2 = 13$

$VarX = EX^2 - (EX)^2 = 13 - 3^2 = 13 - 9 = 4$

$\Pr\{-2 < X < 8\} = \Pr\{-5 < X - 3 < 5\} = \Pr\{|X-3| < 5\} = 1 - \Pr\{|X-3| \geq 5\}$ (1)

$\Pr\{|X-EX| \geq t\} \leq \frac{VarX}{t^2} \Rightarrow \Pr\{|X-3| \geq 5\} \leq \frac{4}{25} \Rightarrow$

$1 - \Pr\{|X-3| \geq 5\} \geq 1 - \frac{4}{25} = \frac{21}{25} \Rightarrow \boxed{\Pr\{-2 < X < 8\} \geq \frac{21}{25}}$

Exercise 7 - Exponential

Thursday, 7 December 2023 01:20

The time (in hours) required to repair a machine is exponentially distributed with parameter λ=0.5. Find:
(a) the probability that the repair time for a machine will exceed 2 hours.
(b) the probability that the repair time for a machine will be at most 10 hours, if we know that it has already exceeded 9 hours.

Let X counting the time until the repair

$X \sim \text{Exponential}(\lambda = 0.5)$

$f(x) = \lambda e^{-\lambda x} = 0.5 e^{-0.5x}$

$F(x) = 1 - e^{-\lambda x}$

a) $P(X>2) = 1 - P(X<2) = 1 - F(2) = 1 - (1 - e^{-0.5 \cdot 2}) = e^{-1}$

β) $P(X<10 \mid X>9) = ?$

<u>Memorylessness</u>: Αν $a,b>0$: $P(X>a+b \mid X>a) = P(X>b)$

$b=9$
$a=1$

$P(X<10 \mid X>9) = 1 - P(X \geq 10 \mid X>9) =$

$= 1 - P(X \geq 9+1 \mid X>9) = 1 - P(X \geq 1) = P(X<1) \Rightarrow$

$\Rightarrow \boxed{P(X<10 \mid X>9) = F(1) = 1 - e^{-0.5}}$

Exercise 8 - Central Limit Theorem

Tuesday, November 26, 2024 12:26 PM

We roll a die until the sum of the results exceeds 300. Find the probability that at least 80 rolls will be needed.

Let X_i be the result of the i-th roll

$$\mu_i = \sum_{x=1}^{6} x P(X_i = x) = \frac{1}{6}(1+2+3+4+5+6) \Rightarrow \boxed{\mu_i = 3.5}$$

$$EX_i^2 = \sum_{x=1}^{6} x^2 P(X_i = x) \ \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6} \simeq 15.1667$$

$$Var X_i = EX_i^2 - (EX_i)^2 \simeq 1.7^2 \Rightarrow \boxed{\sigma_i = 1.7}$$

Let X r.v. of the sum of the rolls

$$X = \sum_{i=1}^{y} X_i$$

To exceed 300, 300 : $\boxed{50 < y < 300}$ (1)

$$P(X > 300)$$

Because of the fact that $X_1, \dots, X_y$ have the same expected value and standard deviation, and for a sufficiently large number of rolls y, we can use the:

<u>Central Limit Theorem</u>

$$P(X > 300) = P\left(\frac{X - y \cdot \mu_i}{\sigma_i \cdot \sqrt{y}} > \frac{300 - y \cdot \mu_i}{\sigma_i \cdot \sqrt{y}}\right) =$$

$$= P\left(Z > \frac{300 - 3.5y}{1.7\sqrt{y}}\right) = 1 - P\left(Z \leq \frac{300 - 3.5y}{1.7\sqrt{y}}\right) = 1 - \Phi\left(\frac{300 - 3.5y}{1.7\sqrt{y}}\right)$$

Let Y r.v. counting the number of rolls until we exceed 300

$$P(Y > 80) = 1 - P(Y < 80) = 1 - \sum^{79} P(Y = y) \Rightarrow$$

$$P(Y \geq 80) = 1 - P(Y < 80) = 1 - \sum_{y=51} P(Y = y) \Rightarrow$$

$$\boxed{P(Y \geq 80) = 1 - \sum_{y=51}^{79} \left(1 - \Phi\left(\frac{300 - 3.5y}{1.7\sqrt{y}}\right)\right)}$$