Discrete Distributions

Distribution	pmf	EX	Var X	Explanation
Bernoulli	$f(x;p)$ $P(X=x)=\begin{cases} p, & x=1 \\ 1-p, & x=0 \end{cases}$	p	qp	Experiment with success probability p
Binomial	$f(x;n,p)$ $P(X=x)=\binom{n}{x}p^x q^{n-x}$	np	npq	x successes in n trials
Geometric	$f(x;p)$ $P(X=x)=q^{x-1}p$	$\frac{1}{p}$	$\frac{q}{p}$	Bernoulli trials in a row until the first success
Poisson	$f(x;\lambda)$ $P(X=x)=e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ	An experiment succeeds with average rate λ (e.g., customers/hour) Probability of x successions? (e.g. x customers this hour)
Negative Binomial	$f(x;r,p)$ $P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$	$\frac{r}{p}$	$\frac{rq}{p^2}$	Bernoulli trial in a row until the r-th success
Hypergeometric	$f(x;N,m,n)$ $P(X=x)=\frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$	$\frac{n\cdot m}{N}$	$\frac{n\cdot m(N-m)(N-n)}{N^2(N-1)}$	N balls, m white, N-m black Select n Probability getting x white balls.

- $q=1-p$

Exercise 1

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We toss a fair coin 5 times and the repetitions are independent. Calculate the pdf of the number of heads.

X : r.v. counting the number of heads

$X \sim \text{Binomial}\left(n=5, p=\frac{1}{2}\right)$

$$P(X=x) = \binom{N}{x} p^x \cdot (1-p)^{N-x} = \binom{5}{x}\left(\frac{1}{2}\right)^x \cdot \left(\frac{1}{2}\right)^{5-x} = \binom{5}{x}\left(\frac{1}{2}\right)^5$$

Exercise 2

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Assume this game: The player chooses a number from 1 to 6. He then throws 3 dices. If the chosen number appears i times($i = 1, 2, 3$) then he wins i points. If the number appears no times, he loses 1 point.

Is the game fair; (fair-> the player stays at the same points no matter how many times he plays)

Suppose that the dices are fair and that the results of the dices are independent.

The player chooses a number. Let it be the number j, $1 \leq j \leq 6$

The probability j appears for a dice is $p = \frac{1}{6}$

Prob. of success : $p = \frac{1}{6}$

" failure : $q = \frac{5}{6}$

X : r.v. counting the number of successes in 3 repetitions

$X \sim \text{Binomial}(n=3, p=\frac{1}{6})$

$$P(X=x) = \binom{3}{x}\left(\frac{1}{6}\right)^x \cdot \left(\frac{5}{6}\right)^{3-x}$$

Let Y: r.v. counting the points for 3 repetitions

$$Y = \begin{cases} X, & X = 1, 2, 3 \\ -1, & X = 0 \end{cases}$$

$$EY = \sum_{y \in \{-1,1,2,3\}} y\, P(Y=y) = -1 \cdot P(Y=-1) + 1 \cdot P(Y=1) + 2P(Y=2) + 3 \cdot P(Y=3) =$$

$$= -P(X=0) + P(X=1) + 2P(X=2) + 3P(X=3) =$$

$$= -\binom{3}{0}\left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^3 + \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 + 2\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 + 3\binom{3}{3}\left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^0 \Rightarrow$$

$\Rightarrow \boxed{EY \simeq -0.07} < 0$

In every game, the player loses 0.07 points on average $\rightarrow$ unfair game

Exercise 3

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The transportation of students from Athens to Chania can be done by ship or by plane. 40% of the students travel by plane. If we have 150 students moving from Athens to Chania, calculate the probability that more than 70 students travel by plane.

$n = 150$

$p = 0.4 \Rightarrow q = 1 - p = 1 - 0.4 \rightarrow q = 0.6$

X: r.v. counting the number of students travelling by plane

$X \sim \text{Binomial}(n = 150, p = 0.4)$

$P(X = x) = \binom{150}{x} 0.4^x \, 0.6^{150-x}$

$$P(X > 70) = \sum_{x=71}^{150} \binom{150}{x} 0.4^x \cdot 0.6^{150-x} = 1 - \sum_{x=0}^{70} \binom{150}{x} 0.4^x \cdot 0.6^{150-x} = 1 - P(X \leq 70)$$

We know that a disk manufactured by a company can be failing with probability 0.01 independently from others. A company sells disks in packs of 10 and has money-back guarantee. Specifically, the customer may return a pack if more than 1 failing disks are contained in it. If someone buys 10 packs, what is the probability returning exactly 1 pack?

X: r.v. counting the number of failing disks in a pack of 10

$X \sim \text{Binomial}(n=10, p=0.01)$

$P(X=x) = \binom{10}{x} 0.01^{x} \cdot 0.99^{10-x}$

$P(X>1) = 1 - P(X \leq 1) = 1 - \left[\binom{10}{0} \cdot 0.01^{0} \cdot 0.99^{10} + \binom{10}{1} \cdot 0.01^{1} \cdot 0.99^{9}\right] \Rightarrow$

$P(X>1) = 1 - (0.99^{10} + 0.1 \cdot 0.99^{9}) = 1 - 0.99^{9} \cdot 1.09 \Rightarrow$

$\Rightarrow \boxed{P(X>1) \simeq 0.0043}$

Y: r.v. counting the number of packs returned out of 3 packs or
" " " containing more than 1 failing disk

$Y \sim \text{Binomial}(n=3, p=P(X>1))$

$P(Y=y) = \binom{3}{y} p^{y} \cdot q^{3-y} = \binom{3}{y} \cdot 0.0043^{y} \cdot 0.9957^{3-y}$

$P(Y=1) = \binom{3}{1} \cdot 0.0043 \cdot 0.9957^{2} \simeq 0.013$

A big experiment has constant probability of success equal to 0.4 for each repetition, where each trial is independent from the others. The experiment is repeated until the first success.

a. Let the cost of the first trial be 100.000€, while each trial then costs 130.000€. Which expression gives the total cost in N trials and what is the expected value of the cost?
b. Let the cost of the first trial be 100.000€, while the cost each time is raised by 30.000€. Which expression gives the total cost in N trials and what is the expected value of the cost?

a) X: r.v. counting the number of trials until the first success.

$X \sim \text{Geometric}\,(p=0.4)$

$P(X=x) = q^{x-1} \cdot p = 0.6^{x-1} \cdot 0.4$

$K_N = 100000 + (N-1)\,130000$

$EX = \frac{1}{p} = \frac{1}{0.4} \Rightarrow \boxed{EX = 2.5}$

$K_{EX} = 100000 + (2.5-1)\cdot 130000 \Rightarrow \boxed{K_{EX} = 295000}$

b) k_i : cost of the i-th trial

$k_i = k_{i-1} + 30000 = k_{i-2} + 2\cdot 30000 = k_{i-3} + 3\cdot 30000 = k_1 + (i-1)\cdot 30000$

$k_1 = 100000$

$K_N = \sum_{i=1}^{N} k_i = \sum_{i=1}^{N}\left(k_1 + (i-1)\cdot 30000\right) = \sum_{i=1}^{N} k_1 + \sum_{i=1}^{N} (i-1)\cdot 30000 =$

$= N\cdot k_1 + 30000 \sum_{i=1}^{N} (i-1) \Rightarrow$

$$\boxed{K_N = N \cdot 30000 + 30000\,\frac{N(N-1)}{2}}$$

$$K_{EX} = 2.5 \cdot 10000 + 30000 \cdot \frac{2.5 \cdot 1.5}{2} \Rightarrow \boxed{K_{EX} = 306250}$$

Exercise 6

Monday, November 18, 2024 12:02 PM

In a newspaper, the average number of mistakes per article is 3 when typed by the person A and 4.2 when typed by the person B. If for an article is equally probable that it was typed by either person, calculate the probability that it has no mistakes.

X_A: r.v. counting the number of mistakes when typed by A

X_B: r.v. counting the number of mistakes when typed by B

$$X_A \sim Poisson(\lambda_A = 3) \Rightarrow P(X_A = x) = e^{-3} \cdot \frac{3^x}{x!} \Rightarrow P(X_A = 0) = e^{-3}$$

$$X_B \sim Poisson(\lambda_B = 4.2) \Rightarrow P(X_B = x) = e^{-4.2} \cdot \frac{4.2^x}{x!} \Rightarrow P(X_B = 0) = e^{-4.2}$$

X: r.v. counting the number of mistakes, regardless of who wrote it

A: the article was written by A

$\Rightarrow \bar{A}$: " " " " B

<u>Total Probability Theorem:</u>

$$P(X=0) = P(X=0 \mid A) \cdot P(A) + P(X=0 \mid B) \cdot P(B) =$$

$$= P(X_A = 0) \cdot P(A) + P(X_B = 0) \cdot P(\bar{A}) =$$

$$= \frac{1}{2} \cdot e^{-3} + \frac{1}{2} e^{-4.2} \Rightarrow \boxed{P(X=0) = \frac{1}{2}(e^{-3} + e^{-4.2})} \cong 0.0324$$

Exercise 7

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In a class there are 20 boys and 15 girls. We randomly choose 4 students.

i) What is the probability of getting 3 boys and 1 girl?

ii) If this experiment is repeated 3 times, what is the probability in 2 times out of 3 that we get 3 boys and 1 girl?

i) X: r.v. counting boys in 4 randomly chosen students

$X \sim \text{Hypergeometric}\ (N=35,\ m=20,\ n=4)$

$$P(X=x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}} = \frac{\binom{20}{x}\binom{15}{4-x}}{\binom{35}{4}} \Rightarrow$$

$$\Rightarrow \boxed{P(X=3) = \frac{\binom{20}{3}\binom{15}{1}}{\binom{35}{4}} \simeq 0.327}$$

ii) Y: r.v. counting the number of times we get 3 boys and 1 girl out of 3 times

$Y \sim \text{Binomial}\ (n=3,\ p=P(X=3)) \Rightarrow$

$$\boxed{P(Y=2) = \binom{3}{2} \cdot 0.327^2 \cdot 0.673^1 \simeq 0.216}$$