- **Conditional Probability**

Desired Event — Given Event

$$P(B|A) = \frac{P(A \cdot B)}{P(A)}$$

Probability that B happens, given that A happens

- Sutlle difference between conditional probability and the probability of the intersection:

$P(A \cap B)$: probability of both A and B happening

$P(B|A)$: A happens given that B happens

If the probability of simple events is equal, then:

$$P(A) = \frac{|A|}{|\Omega|}$$

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} \quad \text{vs} \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{|A \cap B|}{\cancel{|\Omega|}}}{\frac{|A|}{\cancel{|\Omega|}}} = \frac{|A \cap B|}{|A|}$$

We limit the sample space

$A \subseteq \Omega \Rightarrow |A| \leq |\Omega|$

- It holds that $P(B|A) = \frac{P(AB)}{P(A)} \Rightarrow \boxed{P(AB) = P(A) \cdot P(B|A)}$

Similarly: $\boxed{P(AB) = P(B) \cdot P(A|B)}$

- **Complementary Conditional Probability:** $\boxed{P(\bar{A}|B) = 1 - P(A|B)}$

- **Stohastic Independence:** A and B stohastically independent events $\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

- $P(A|B) = \frac{P(AB)}{P(B)} \xRightarrow{\text{A, B stohastically independent}} P(A|B) = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} \Rightarrow \boxed{P(A|B) = P(A)}$

A happening does not depend on B happening

- Similarly, if A and B are stohastically independent, $\boxed{P(B|A) = P(B)}$

- **Total Probability Theorem**

If $A_1, \ldots, A_n$ partition of Ω, then

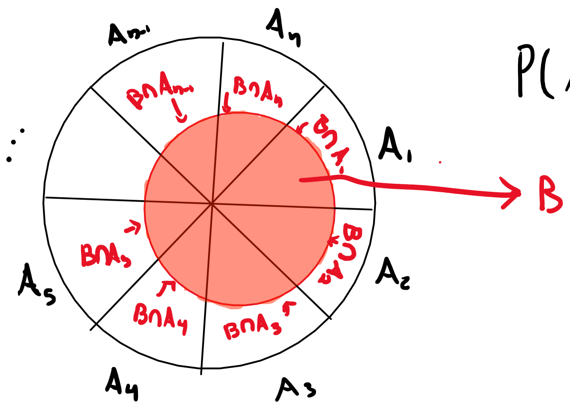
Total Probability Theorem

If $A_1, \dots, A_n$ partition of Ω, then

$$\boxed{P(B) = \sum_{k=1}^{n} P(A_k)P(B|A_k)}$$

$\left(\text{because } P(B) = \sum_{k=1}^{n} P(A_k \cap B)\right)$

$$P(A_k \cap B) = P(A_k) \cdot P(B|A_k)$$



• Bayes Theorem

Let H: Cause (hypothesis) E: Result (Event)

Then, the probability $P(H|E)$ expresses the probability:

- The result is due to the cause

alternative

How probable is the cause given the fulfilment of the event?

Knowledge of this probability is crucial in many applications!

Bayes Theorem: $\boxed{P(H|E) = \dfrac{P(H) \cdot P(E|H)}{P(E)}}$

Example:

Let H: disease, E: positive test. Then:

prior probability

we assume that these are known or easy to calculate

(A pharmaceutical company can obtain this data through statistical analysis)

$P(H)$: probability someone having the disease

$P(E)$: probability someone having positive test

$P(E|H)$: probability someone having positive test, given that he has the disease

$P(H|E)$: probability someone having the disease given that he has a positive test. ← posterior probability

Desired, but hard to calculate

given that he has a positive test. posterior probability

Desired, but hard to calculate independently

The desired probability can be calculated using Bayes theorem!!!

Exercise 1

Tuesday, October 22, 2024 1:35 PM

A farmer has a 10 trees in a row.

a. What is the probability exactly 4 trees getting sick?
b. What is the probability exactly 4 trees getting sick and that they are in a row?
c. If exactly 4 trees got sick, what is the probability of these trees being in a row?

a) $|\Omega| = 2^{10}$

A: Exactly 4 trees get sick

$|A| = \binom{10}{4}$

$P(A) = \dfrac{\binom{10}{4}}{2^{10}}$

β) Β: Trees in a row get sick

$|AB| = 7 \Rightarrow P(AB) = \dfrac{|AB|}{|\Omega|} = \dfrac{7}{2^{10}}$

γ) $P(B|A) = \dfrac{P(AB)}{P(A)} = \dfrac{\frac{7}{\cancel{2^{10}}}}{\frac{\binom{10}{4}}{\cancel{2^{10}}}} \Rightarrow \boxed{P(B|A) = \dfrac{7}{\binom{10}{4}}}$

Exercise 2

Tuesday, October 22, 2024 5:16 PM

An insurance company divides its customers into two categories; those who are prone to accidents and those who are not. Through statistical analysis, it is known that someone prone to accidents will have an accident in a year with probability 0.4, while someone not prone to accidents will have an accident in a year with probability 0.2.

A) Supposing that 30% ot the population is prone to accidents, then what is the probability a new customer having an accident in the next year?
B) If a customer had an accident the first year, what is the probability that he is prone to accidents?

H: prone to accidents
E: has an accident this year

$P(E|H) = 0.4$

$P(E|\bar{H}) = 0.2$

A) $P(H) = 0.3 \Rightarrow P(\bar{H}) = 0.7$ $H, \bar{H} \rightarrow$ partition

$P(E) = ?$

Total Probability T.: $P(E) = P(H) \cdot P(E|H) + P(\bar{H}) \cdot P(E|\bar{H}) =$

$= 0.3 \cdot 0.4 + 0.7 \cdot 0.2 \Rightarrow$

$\Rightarrow \boxed{P(E) = 0.26}$

B) $P(H|E) = ?$

Bayes T.: $P(H|E) = \dfrac{P(H) \cdot P(E|H)}{P(E)} = \dfrac{0.3 \cdot 0.4}{0.26} \Rightarrow \boxed{P(H|E) \simeq 0.46}$

Exercise 3

Tuesday, October 22, 2024 5:39 PM

A factory has two machine A and B, manufacturing 60% and 40% of the total production items in respect. The percentage of failing products is 3% for the machine A and 5% for the machine B. Find the probability that a failing product was manufactured by machine B.

E: failing product

H: produced by A

produced by B

$P(H) = 0.6 \Rightarrow P(\bar{H}) = 0.4$

$P(E|H) = 0.03$

$P(E|\bar{H}) = 0.05$

$P(\bar{H}|E) = ?$

<u>Total Probability T.</u>: $P(E) = P(E|H)P(H) + P(E|\bar{H}) \cdot P(\bar{H}) =$

$= 0.03 \cdot 0.6 + 0.05 \cdot 0.4 \Rightarrow$

$\Rightarrow \boxed{P(E) = 0.038}$

<u>Bayes T.</u>: $P(\bar{H}|E) = \dfrac{P(E|\bar{H}) \cdot P(\bar{H})}{P(E)} = \dfrac{0.05 \cdot 0.4}{0.038} \Rightarrow \boxed{P(\bar{H}|E) \simeq 0.53}$

Exercise 4

Tuesday, October 22, 2024 6:24 PM

Let two boxes, where the first one contains a white ball and a black ball, while the second one contains two black balls and a white ball. We randomly choose a box and then we randomly get a ball out of it.

A) What is the probability the ball drawn is black?
B) If the ball drawn is black, what is the probability that it was drawn by the first box?

A: Box 1 is chosen

B: The ball drawn is black

Box 2 is chosen

$P(A) = \frac{1}{2} \Rightarrow P(\bar{A}) = \frac{1}{2}$

$P(B|A) = \frac{1}{2} \Rightarrow P(\bar{B}|A) = \frac{1}{2}$

$P(B|\bar{A}) = \frac{2}{3} \Rightarrow P(\bar{B}|\bar{A}) = \frac{1}{3}$

A) Total Probability T.: $P(B) = P(A) \cdot P(B|A) + P(\bar{A}) \cdot P(B|\bar{A}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} \Rightarrow$

$\Rightarrow \boxed{P(B) = \frac{7}{12}}$

B) Bayes T.: $P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)} = \dfrac{\frac{1}{\cancel{2}} \cdot \frac{1}{\cancel{2}}}{\frac{7}{\cancel{12}_{3}}} \Rightarrow \boxed{P(A|B) = \frac{3}{7}}$

Exercise 5

Tuesday, October 22, 2024 6:02 PM

A family has i children with probability P_i, where $P_1 = 0.1$, $P_2 = 0.25$, $P_3 = 0.35$ and $P_4 = 0.3$.
A child is randomly chosen from this family. If the child is the oldest, what is the probability that the family has:
A) a single child
B) 4 children
Assume there cannot exist twin children.

A_1: 1 child

A_2: 2 children

A_3: 3 children

A_4: 4 children

B: the oldest child is chosen

$P(A_1) = 0.1$

$P(A_2) = 0.25$

$P(A_3) = 0.35$

$P(A_4) = 0.3$

$P(B|A_1) = 1$

$P(B|A_2) = 0.5$

$P(B|A_3) = 0.333\ldots$

$P(B|A_4) = 0.25$

A) $P(A_1|B) = ?$

$A_1, A_2, A_3, A_4 \rightarrow$ partition of Ω

Total Probability T.: $P(B) = \sum_{i=1}^{4} P(B|A_i) \cdot P(A_i) =$

$= 0.1 \cdot 1 + 0.25 \cdot 0.5 + 0.35 \cdot 0.333\ldots + 0.3 \cdot 0.25 \Rightarrow$

$\Rightarrow \boxed{P(B) \simeq 0.4167}$

Bayes T.: $P(A_1|B) = \dfrac{P(B|A_1) \cdot P(A_1)}{P(B)} = \dfrac{1 \cdot 0.1}{0.4167} \Rightarrow \boxed{P(A_1|B) \simeq 0.24}$

B) **Bayes T.**: $P(A_4|B) = \dfrac{P(B|A_4) \cdot P(A_4)}{P(B)} = \dfrac{0.25 \cdot 0.3}{0.4167} \Rightarrow$

$$\Rightarrow \boxed{P(A_4 | B) \simeq 0.18}$$

Exercise 6 (Belief update)

Wednesday, October 23, 2024 2:11 PM

A pharmaceutical company produces a test for a disease, which has probability of getting positive on someone if he has the disease equal to 99%, while it gives positive test on someone who does not have the disease with probability 2%.

A) If 0.1% of the population may have this disease, then what is the probability that someone has the disease, given that he has a positive test?
B) Given that someone has a positive test, repeats the test and it gives a positive test again, then what is the probability that he has the disease?

A) H: disease

E: positive test

$P(E|H) = 0.99$

$P(E|\bar{H}) = 0.02$

Initially, we do not have any prior knowledge of the person having the disease. So, it is safe enough to use the probability drawn by statistical analysis ⇒

⇒ $P(H) = 0.001$

$P(H|E) = ?$

Total Probability Theorem: $P(E) = P(H) \cdot P(E|H) + P(\bar{H}) \cdot P(E|\bar{H}) =$

$= 0.001 \cdot 0.99 + 0.999 \cdot 0.02 \Rightarrow$

$\Rightarrow P(E) = 0.02097$

Bayes T.: $P(H|E) = \dfrac{P(H)\,P(E|H)}{P(E)} = \dfrac{0.001 \cdot 0.99}{0.02097} \Rightarrow$

$\Rightarrow \boxed{P(H|E) \simeq 0.047 \quad ή \quad P(H|E) \simeq 4.7\%}$

B) At this point, we have some prior knowledge of the person having the disease. So, we can use the probability calculated as prior ⇒

⇒ $P(H) = 0.047 \Rightarrow P(\bar{H}) = 0.953$

Total Probability T. $P(E) = P(H) \cdot P(E|H) + P(\bar{H}) \cdot P(E|\bar{H}) =$

$= 0.047 \cdot 0.99 + 0.953 \cdot 0.01$

$\Rightarrow P(E) = 0.05606$

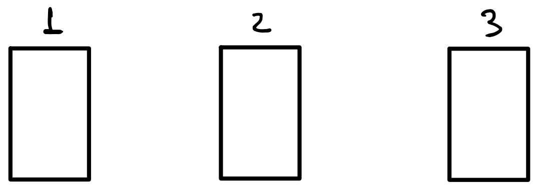
<u>Bayes T.</u> : $P(H|E) = \dfrac{P(E|H) \cdot P(H)}{P(E)} = \dfrac{0.99 \cdot 0.047}{0.05606} \Rightarrow$

$P(H|E) \simeq 0.83 \Rightarrow \boxed{P(H|E) = 83\ \%}$

Exercise 7 (Monty Hall)*

Wednesday, October 23, 2024 2:28 PM

In a TV show, there is the host and the player. There are 3 doors, from which only one door has the price and the other 2 have nothing. The first time, the host requests from the player to choose one door. After the player chooses a door, the host opens a door containing nothing and asks the player if he keeps the initial door or changes the door. Is there a better choice for the player and, if yes, what would this choice be?



Lets define the following events:

- A: I choose the correct door the first time
- B: I change the door the second time
- C: I win the price

We actually want to compare the probabilities $P(C|B)$ and $P(C|\bar{B})$

It holds that: $P(A) = \frac{1}{3} \Rightarrow P(\bar{A}) = \frac{2}{3}$

$P(B) = \frac{1}{2} \Rightarrow P(\bar{B}) = \frac{1}{2}$ (change door or not randomly)

- $P\{C|(A \cap B)\}$: Probability winning given that I chose correct the first time and change door. Given this sequence of events, if I chose correct the first time, the presenter will open a wrong door and the doors remaining will be the correct, that I already chose, and the other wrong door. Then, I change door and lose, so:

$$\boxed{P\{C|(A \cap B)\} = 0} \quad (1)$$

- $P\{C|(A \cap \bar{B})\}$: Probability winning given that I chose correct the first time and don't change door. Given this sequence of events, if I chose correct the first time, the presenter will open a wrong door and the doors remaining will be the correct, that I already chose, and the other wrong door. Then, I don't change door and win, so:

$$\boxed{P\{C|(A \cap \bar{B})\} = 1} \quad (2)$$

• $P\{C|(\bar{A}\cap B)\}$: Probability winning given that I chose wrong the first time and change door. Given this sequence of events, if I chose wrong the first time, the presenter will open the other wrong door and the doors remaining will be the wrong, that I already chose, and the correct door. Then, I change door and win, so:

$$\boxed{P\{C|(\bar{A}\cap B)\}=1} \quad (3)$$

• $P\{C|(\bar{A}\cap \bar{B})\}$: Probability winning given that I chose wrong the first time and don't change door. Given this sequence of events, if I chose wrong the first time, the presenter will open the other wrong door and the doors remaining will be the wrong, that I already chose, and the correct door. Then, I don't change door and lose, so:

$$\boxed{P\{C|(\bar{A}\cap \bar{B})\}=0} \quad (4)$$

A and $\bar{A}$ partition $\xRightarrow{\text{Total Probability Theorem}}$

$$\Rightarrow \begin{cases} P(BC)=P(ABC)+P(\bar{A}BC)=\cancel{P(C|AB)}^{0}P(AB)+\cancel{P(C|\bar{A}B)}^{1}\cdot P(\bar{A}B) \Rightarrow \boxed{P(BC)=P(\bar{A}B)} \quad (5) \\ P(\bar{B}C)=P(A\bar{B}C)+P(\bar{A}\bar{B}C)=\cancel{P(C|A\bar{B})}^{1}\cdot P(A\bar{B})+\cancel{P(C|\bar{A}\bar{B})}^{0}\cdot P(\bar{A}\bar{B}) \Rightarrow \boxed{P(\bar{B}C)=P(A\bar{B})} \quad (6) \end{cases}$$

But, A and B independent events, so:

$$(5) \Rightarrow P(BC)=P(\bar{A})\cdot P(B) \Rightarrow P(C|B)\cdot\cancel{P(B)}=P(\bar{A})\cdot\cancel{P(B)} \Rightarrow \boxed{P(C|B)=\frac{2}{3}}$$

$$(6) \Rightarrow P(\bar{B}C)=P(A)\cdot P(\bar{B}) \Rightarrow P(C|\bar{B})\cdot P(\bar{B})=P(A)\cdot P(\bar{B}) \Rightarrow \boxed{P(C|\bar{B})=\frac{1}{3}}$$

Thus, $P(C|B)=\frac{2}{3}>\frac{1}{3}=P(C|\bar{B}) \Rightarrow$ Best strategy is to change door!

Intuition: It is better that I change door, because it is more probable that I chose a wrong door initially. Because of this, changing the door will always give the correct door that I chose a wrong door initially. Because of this, changing the door will always give the correct door.