Theory

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Let the distribution of the population and Θ an unknown parameter which we want to estimate via a sample

- Estimator: Any function with which we want to estimate the value of Θ:
  - dependent to the sample
  - independent to Θ

- Criteria to evaluate estimators
  - Unbiased: $ET = \Theta$ (validity)
  - Effectiveness: $VarT\downarrow$ (if $VarT_1 < VarT_2 \Rightarrow T_1$ more effective) (reliability)
  - Consistency: If $T_n \xrightarrow{P} \Theta$ while $n \to \infty$

    ↳ If T_n unbiased and $\lim_{n\to\infty} VarT_n \to 0 \Rightarrow T_n$ consistent

- Methods getting estimators
  - Moments method
  - Maximum likelihood method
  - Least Squares Method → Regression

Exercise 1

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Let T_1 and T_2 unbiased estimtors and θ an unknown parameter. Let also the estimaror $T = \lambda T_1 + (1-\lambda)T_2$, where λ is constant. Is T unbiased?

Unbiased estimator T $\Rightarrow$ $ET = \theta$

$ET_1 = \theta$

$ET_2 = \theta$

linearity

$$ET = E[\lambda T_1 + (1-\lambda)T_2] = \lambda ET_1 + (1-\lambda)ET_2 =$$

$$= \lambda\theta + (1-\lambda)\theta \Rightarrow ET = \theta \Rightarrow$$

$\Rightarrow T$ unbiased

Exercise 2

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Let a random sample $X_1, X_2, \dots, X_n$ drawn from a distribution with an unknown parameter θ, and let the estimator: $T = \sum_{i=1}^{n} \lambda_i X_i$, where λ_i constant $(i = 1, 2, \dots, n)$. What must hold for the constants λ_i so that T be unbiased for the unknown parameter θ;

linearity

$$ET = \theta \Rightarrow E\left[\sum_{i=1}^{n} \lambda_i X_i\right] = \theta \Rightarrow \sum_{i=1}^{n} \lambda_i EX_i = \theta \Rightarrow \sum_{i=1}^{n} \lambda_i \mu = \theta \Rightarrow$$

$$\Rightarrow \mu \sum_{i=1}^{n} \lambda_i = \theta \Rightarrow \boxed{\sum_{i=1}^{n} \lambda_i = \frac{\theta}{\mu}}$$

Exercise 3

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Let $X_1, X_2, \dots, X_n$ random sample of a population with distribution for which it holds $\mu = \theta$ και $\sigma^2 = 2\theta$.

Show that the estimators $T_1 = \frac{X_1 + X_2}{2}, T_2 = \frac{3X_1 + 2X_2}{5}$ are unbiased for θ and find the most effective.

$$ET_1 = E\left[\frac{X_1 + X_2}{2}\right] = \frac{EX_1 + EX_2}{2} = \frac{\theta + \theta}{2} = \theta \Rightarrow \text{unbiased}$$

$$ET_2 = E\left[\frac{3X_1 + 2X_2}{5}\right] = \frac{3EX_1 + 2EX_2}{5} = \frac{3\theta + 2\theta}{5} = \theta \Rightarrow \text{unbiased}$$

$$VarT_1 = Var\left[\frac{X_1 + X_2}{2}\right] = \frac{1}{2^2}\left(VarX_1 + VarX_2\right) = \frac{1}{4}(2\theta + 2\theta) = \theta$$

$$VarT_2 = Var\left[\frac{3X_1 + 2X_2}{5}\right] = \frac{1}{5^2}\left(3^2 VarX_1 + 2^2 VarX_2\right) = \frac{1}{25}\left(9 \cdot 2\theta + 4 \cdot 2\theta\right) \Rightarrow$$

$$\Rightarrow VarT_2 = \frac{26}{25}\theta > \theta = VarT_1 \Rightarrow T_1 \text{ more effective}$$

Exercise 4

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Let $X_1, X_2, \dots, X_\nu$ and random sample and p=P(X≤a). Show that the estimator $T(X_1, X_2, \dots, X_\nu) = \frac{1}{\nu}\left[number\ of\ X_\kappa, \kappa = 1,2,..,\nu\ so\ that\ X_\kappa \le \alpha\right)$ is unbiased.

Let Y_k r.v. so that $EY_k = p$

$$Y_k = \begin{cases} 1, & X_k \le a \\ 0, & X_k > a \end{cases} \qquad P(Y_k = 1) = p \Rightarrow P(Y_k = 0) = 1-p$$

Y r.v. so that $Y = \sum_{i=1}^{\nu} Y_i$ → indicates how many $X_k \le a$

Thus, $T(X_1, \dots, X_\nu) = \frac{1}{\nu} Y \Rightarrow ET(X_1, \dots, X_\nu) = E\left[\frac{Y}{\nu}\right] = \frac{EY}{\nu} =$

$= \frac{1}{\nu} E\left[\sum_{i=1}^{N} Y_i\right] = \frac{1}{\nu} \sum_{i=1}^{\nu} EY_i = \frac{1}{\nu} \sum_{i=1}^{\nu} p = \frac{1}{\cancel{\nu}} \cancel{\nu} \cdot p \Rightarrow ET(X_1, \dots, X_n) = p \Rightarrow$

$\Rightarrow T(X_1, \dots, X_n)$ unbiased

Exercise 5

Thursday, December 19, 2024 10:42 AM

Let $X_1, X_2, \ldots, X_n$ random sample from a distribution with p.d.f. $f(x) = \theta x^{\theta-1},\ 0 < x < 1,\ \theta > 0$. Find and estimator for θ using:

(α) the moments method,

(β) with the maximum likelihood estimator method

Έχουμε μία μόνο άγνωστη παράμετρο, άρα θα χρησιμοποιήσουμε την 1η ροπή, δηλαδή τη μέση τιμή:

α) Moments method

1st step: There is only one unknown parameter, so we will use the 1st moment, i.e. the mean value:

$$EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\theta x^{\theta-1}\,dx = \int_0^1 \theta x^{\theta}\,dx = \theta \int_0^1 x^{\theta}\,dx = \theta\left[\frac{x^{\theta+1}}{\theta+1}\right]_0^1 =$$

$$= \theta\left(\frac{1^{\theta+1}}{\theta+1} - \frac{0^{\theta+1}}{\theta+1}\right) \Rightarrow \boxed{EX = \frac{\theta}{\theta+1}}$$

2nd step: Let $EX = \bar{X} \Rightarrow \dfrac{\theta}{\theta+1} = \bar{X} \Rightarrow$

$$\Rightarrow \theta = \theta\bar{X} + \bar{X} \Rightarrow \theta(1-\bar{X}) = \bar{X} \Rightarrow \boxed{\theta = \frac{\bar{X}}{1-\bar{X}}}$$

β) Maximum Likelihood Estimator Method

$$L(\theta) = f(x_1, x_2, \ldots, x_n; \theta) = f(x_1;\theta)\cdot f(x_2;\theta)\cdot \ldots \cdot f(x_n;\theta) \Rightarrow$$

$$\log L(\theta) = \log\left[f(x_1;\theta)\cdot f(x_2;\theta) \ldots \cdot f(x_n;\theta)\right] = \sum_{i=1}^{n} \log f(x_i;\theta) =$$

$$= \sum_{i=1}^{n} \log\left(\theta x_i^{\theta-1}\right) = \sum_{i=1}^{n}\left(\log\theta + \log x_i^{\theta-1}\right) = \sum_{i=1}^{n}\left(\log\theta + (\theta-1)\log x_i\right) =$$

$$= \sum_{i=1}^{n}\log\theta + \sum_{i=1}^{n}(\theta-1)\log x_i = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i \Rightarrow$$

$$\frac{d\log L(\theta)}{d\theta} = \frac{d\left(n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i\right)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i$$

$$\left.\frac{d\log L(\theta)}{d\theta}\right|_{\theta=\hat{\theta}} = 0 \Rightarrow \frac{n}{\hat{\theta}} + \sum_{i=1}^{n}\log x_i = 0 \Rightarrow \hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \ldots} \quad (1)$$

$$\left.\frac{d \log L(\theta)}{d\theta}\right|_{\theta=\hat{\theta}} = 0 \Rightarrow \frac{n}{\hat{\theta}} + \sum_{i=1}^{n} \log x_i = 0 \Rightarrow \boxed{\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}} \quad (1)$$

Θέλουμε να δείξουμε ότι $\frac{d^2 \log L(\theta)}{d\theta^2} < 0$

$$\frac{d^2 \log L(\theta)}{d\theta^2} = \frac{d\left(\frac{n}{\theta} + \sum_{i=1}^{n} \log x_i\right)}{d\theta} = -\frac{n}{\theta^2} < 0$$

Thus, $L(\theta)$ maximizes when $\boxed{\hat{\theta} = -\frac{n}{\sum_{i=1}^{n} \log x_i}}$