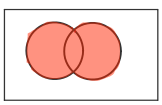
Basic Definitions

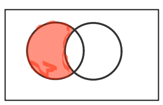
- Random Experiment: A priori unknown outcome
- Sample Space: Set of all possible outcomes. We use the symbol Ω for the sample space
- Sample Point: A single element of the sample space
- Event:
  - Set of sample points
  - Subset of the sample space
- Occurrence of event: the outcome of an experiment (a sample point) is contained in the event

Events ⟷ Sets, so we use set operations

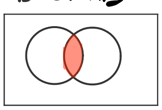
- $A \cup B$ → Union



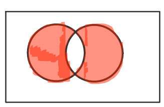
- $A - B = A \cap \bar{B} = A\bar{B}$ → Difference: A occurs but not B



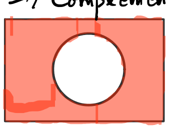
- $A \cap B$ or AB → Intersection



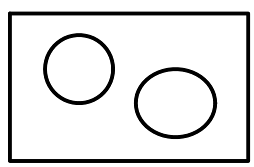
- $A \oplus B = (A-B) \cup (B-A) = A\bar{B} \cup \bar{B}A$: Only A occurs or only B occurs



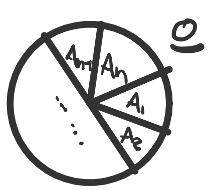
- $\bar{A}$ → Complement



- A and B are disjoint ⟺ $A \cap B = \emptyset$



- $A_1, A_2, \dots, A_n$ partition of Ω ⟺
$$\begin{cases} \forall i \neq j \quad A_i \cap A_j = \emptyset \\ \bigcup_i A_i = \Omega \end{cases}$$



- De Morgan's Law
  - For 2 sets A_1 and A_2:
$$\overline{A_1 A_2} = \bar{A_1} \cup \bar{A_2}$$
$$\overline{A_1 \cup A_2} = \bar{A_1}\,\bar{A_2}$$
  - For n sets $A_1, A_2, \dots, A_n$
$$\overline{A_1 A_2 \dots A_n} = \bar{A_1} \cup \bar{A_2} \cup \dots \cup \bar{A}_{n-1} \cup \bar{A_n} = \bigcup_{i=1}^{n} \bar{A_i}$$
$$\overline{A_1 \cup A_2 \cup \dots \cup A_{n-1} \cup A_n} = \bar{A_1}\,\bar{A_2} \dots \bar{A}_{n-1}\,\bar{A_n} = \bigcap_{i=1}^{n} \bar{A_i}$$

- Probability definitions
  - Classic Probability: (De Moivre, Laplace)

    If A is an event:

    $$P(A)=\frac{N(A)}{N}=\frac{|A|}{|\Omega|}=\frac{\#\text{ of favorable outcomes for event } A}{\#\text{ of all outcomes}}$$

    $N(A)=|A|$: cardinality of A → number of sample points contained in A

    Requirements:
    - Finete sample space
    - Equally likely outcomes
  - Limit of relative frequency

    $$P(A)=\lim_{N\to\infty}\frac{N(A)}{N}$$

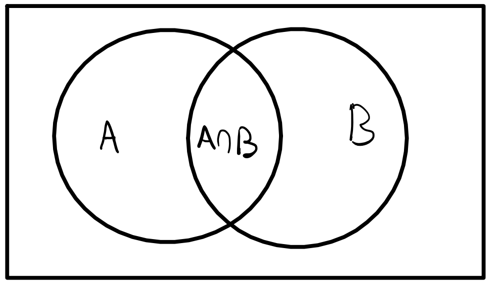
    - Convergence to a single value after many repetitions
    - Empirical (statistical) calculation of probability
  - Axiomatic Foundation (Kolmogorov)

    Probability is a set function of subsets of Ω to real numbers

    Axioms
    - Axiom 1: $P(A)\geq 0$
    - Axiom 2: $P(\Omega)=1$
    - Axiom 3: If $A\cap B=\emptyset \Rightarrow Pr\{A\cup B\}=Pr\{A\}+Pr\{B\}$

Properties

- Property 1: $0\leq P(A)\leq 1$
- Property 2: $P(A)+P(\bar{A})=P(\Omega)=1$
- Property 3: $Pr\{A\cup B\}=Pr\{A\}+Pr\{B\}-Pr\{A\cap B\}$



- **Property 4:** (Boole's Inequality): $P\{\bigcup_i A_i\} \leq \sum P(A_i)$

$$P(A \cup B) \leq P(A) + P(B)$$

↓

$$P(A \cup B) = P(A) + P(B)$$ → Holds iff $A \cap B = \emptyset$

Theory - Example

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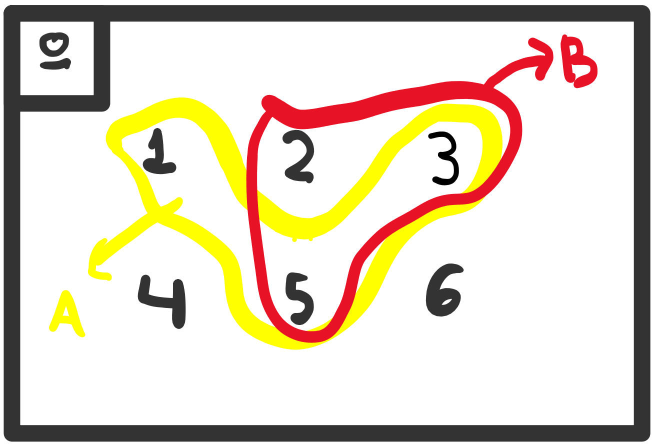
We roll a fair dice.
The sample space is Ω = {1,2,3,4,5,6}
A = {"Outcome is an odd number"}
B = {"Outcome is a prime number"}

1) Which are the elements of A?
2) Which are the elements of A?
3) Which are the elements of the event "both events happen"?
4) Which are the elements of the event "at least one event happens"?
5) What is the probability of (3)?
6) What is the probability of (4)?



1) $A = \{1,3,5\}$

2) $B = \{2,3,5\}$

3) $\{3,5\} = AB$

4) $A \cup B = \{1,2,3,5\}$

5) $P(AB) = \dfrac{|AB|}{|\Omega|} = \dfrac{2}{6} = \dfrac{1}{3}$

6) $P(A \cup B) = \dfrac{|A \cup B|}{|\Omega|} = \dfrac{4}{6} = \dfrac{2}{3}$

Exercise 1

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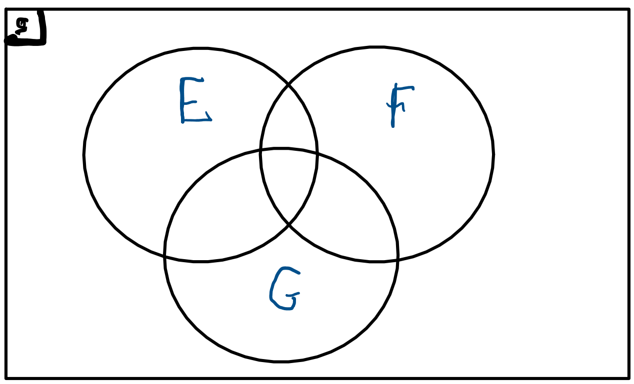
Let E, F and G be 3 sets. Give the expression for each of the following events:

α	Only E occurs	$E\bar{F}\bar{G}$
β	E and F occur, but not G	$EF\bar{G}$
γ	At least 2 events occur	$EF\bar{G} \cup E\bar{F}G \cup \bar{E}FG \cup EFG$ ή $EF \cup EG \cup FG$
δ	All 3 events occur:	EFG
ε	None of the three events occur	$\bar{E}\bar{F}\bar{G}$
στ	At least one event occurs	$E \cup F \cup G$
ζ	Maximum one event occurs	$\overline{EF \cup EG \cup FG}$
η	Maximum 2 events occur	$\overline{EFG}$
θ	Exactly 2 events occur	$EF\bar{G} \cup E\bar{F}G \cup \bar{E}FG$
ι	Maximum 3 events occur	Ω



Exercise 2

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We roll 2 similar dices and assume the following events:

- E: the sum is an odd number
- F: at least a dice outcome is '1'
- G: the sum is equal to 5

Give the possible (favorable) outcomes of the following events:

α) $EF = \{\{1,2\},\{1,4\},\{1,6\}\}$

β) $E \cup F = \{\{1,1\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,5\},\{4,3\},\{4,5\},\{6,3\},\{6,5\}\}$

γ) $FG = \{\{1,4\}\}$

δ) $EFG = \{\{1,4\}\}$

στ) $F\bar{G} = \{\{1,1\},\{1,2\},\{1,3\},\{1,5\},\{1,6\}\}$

Exercise 3

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Let 'RESERVE' and 'VERTICAL' be 2 words. A letter from each word is chosen. What is the probability of choosing the same letter?

$|\Omega| = 7 \cdot 8 = 56$

Favorable outcomes:

- Letter E is chosen from both: $|E| = 3 \cdot 1 = 3$
- " R " " " : $|R| = 2 \cdot 1 = 2$
- " V " " " : $|V| = 1 \cdot 1 = 1$

$|E \cup R \cup V| = 3 + 2 + 1 = 6$

$P(E \cup R \cup V) = \frac{|E \cup R \cup V|}{|\Omega|} = \frac{6}{56}$

$P(E \cup V \cup R) = P(E) + P(V) + P(R)$

because $E \cap V = \emptyset$ and $E \cap R = \emptyset$ and $V \cap R = \emptyset$

Exercise 4

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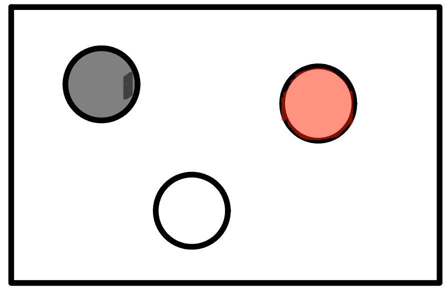
Suppose we have a box containing 3 balls, and red one, a black one, and a white one. A ball is drawn, then it is placed back, and then a ball is drawn again.

a) What is the probability of drawing the same color both times?
b) What is the probability of drawing the black ball any of the two phases?



M : black
K : red
A : white

$$\Omega = \{(MM), (MK), (MA), (KM), (KK), (KA), (AM), (AK), (AA)\} \Rightarrow |\Omega| = 9$$

a) A: I draw the same color (same ball) at both phases:

$$A = \{(MM), (KK), (AA)\} \Rightarrow |A| = 3$$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{3}{9} = \frac{1}{3}$$

b) B: I draw the black ball at any of the two phases

$$B = \{(MM), (MA), (MK), (AM), (KM)\} \Rightarrow$$

$$\Rightarrow |B| = 5 \Rightarrow P(B) = \frac{|B|}{|\Omega|} = \frac{5}{9}$$

Exercise 5

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Let E and F be two events, where:

- P(E) = 0.9
- P(F) = 0.8

Prove that: P(EF) >= 0.7

$P(E \cup F) = P(E) + P(F) - P(EF)$

$0 \leq P(E \cup F) \leq 1 \Rightarrow 0 \leq P(E) + P(F) - P(EF) \leq 1 \Rightarrow$

$P(E) + P(F) - P(EF) \leq 1 \Rightarrow 0.9 + 0.8 - P(EF) \leq 1 \Rightarrow$

$1.7 - P(EF) \leq 1 \Rightarrow 1.7 - 1 \leq P(EF) \Rightarrow \boxed{P(EF) \geq 0.7}$

Exercise 6

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Let:

- A: "The temperature in Patras is 25C"
- B: "The temperature in Thessaloniki is 25C"
- C: "The highest temperature between the two cities is 25C"

If P(A) = 0.3, P(B) = 0.4 and P(C) = 0.2, calculate the probability the lowest temperature between the two cities is 25C.

Let t_1, t_2 be the temperatures in Patras and Thessaloniki in respect

A: $t_1 = 25$ and $t_2 \in \mathbb{R}$

B: $t_2 = 25$ and $t_1 \in \mathbb{R}$

C: $\max(t_1, t_2) = 25 \Rightarrow \begin{cases} t_1 = 25 \text{ and } t_2 \leq 25 \\ t_2 = 25 \text{ and } t_1 \leq 25 \end{cases}$

Let D: the minimum temperature between the two cities be 25C°

D: $\min(t_1, t_2) = 25 \Rightarrow \begin{cases} t_1 = 25 \text{ και } t_2 \geq 25 \\ t_2 = 25 \text{ και } t_1 \geq 25 \end{cases}$

AB: $t_1 = 25$ and $t_2 = 25$
CD: $t_1 = 25$ and $t_2 = 25$
$\Rightarrow \boxed{AB = CD}$ (1)

A∪B: $t_1 = 25$ or/and $t_2 = 25$
C∪D: $t_1 = 25$ or/and $t_2 = 25$
$\Rightarrow \boxed{A \cup B = C \cup D}$ (2)

$(2) \Rightarrow P(A \cup B) = P(C \cup D) \Rightarrow P(A) + P(B) - \cancel{P(AB)} = P(C) + P(D) - \cancel{P(CD)} \overset{(1)}{\Rightarrow}$

$\Rightarrow P(A) + P(B) = P(C) + P(D) \Rightarrow P(D) = P(A) + P(B) - P(C) = 0.3 + 0.4 - 0.2 \Rightarrow$

$\Rightarrow \boxed{P(D) = 0.5}$