ΝΑΝΟΗΛΕΚΤΡΟΝΙΚΗ & ΚΒΑΝΤΙΚΕΣ ΠΥΛΕΣ

$5^η$ Διάλεξη

Βιβλιογραφία: EXPLORATIONS IN QUANTUM COMPUTING, Colin P. Williams (2nd edition, Springer-Verlag, 2011), chapter 1.

Extracting Answers from Quantum Computers

- Observables in Quantum Mechanics
- Observing in the Computational Basis
- Alternative Bases
- Change of Basis
- Observing in an Arbitrary Basis
- Problems with solutions

Observing in the Computational Basis

A n-qubit register can be found in a superposition of all the 2^n possible ***bit strings***:

$$|00\ldots0\rangle, |00\ldots1\rangle, \ldots, |11\ldots1\rangle$$

When we describe the state of a multi-qubit quantum memory register as a superposition of its possible bit-string configurations, we say the state is represented in the ***computational basis***.

The most common measurement in quantum computing is to measure a set of qubits "*in the computational basis*", where the *spin orientation* of each qubit in the quantum memory register is measured along an axis *parallel to the z-axis of the Bloch sphere*, which is the axis passing through North and South poles.

Each qubit will be found to be aligned or anti-aligned with the z-axis, with "spin-up" (i.e., in state $|0\rangle$) or "spin-down" (i.e., in state $|1\rangle$) respectively.

Complete Readout

Consider a 3-qubit quantum memory register that initially is in the state:

$$|\psi\rangle = c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle$$

where $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 + |c_5|^2 + |c_6|^2 + |c_7|^2 = 1$

For convenience imagine labeling the leftmost qubit A, the middle qubit B, and the rightmost qubit C. The probabilities of obtaining the eight distinct triples of values when three qubits are read in the computational basis are:

Qubit A	Qubit B	Qubit C	Probability
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|c_0\|^2$
$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|c_1\|^2$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|c_2\|^2$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|c_3\|^2$
$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|c_4\|^2$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|c_5\|^2$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|c_6\|^2$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|c_7\|^2$

Partial Readout

Consider again the 3-qubit quantum memory register that initially is in the state:

$$|\psi\rangle = c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle$$

Suppose we measure only the middle qubit, B, and find it to be in state $|1\rangle$

This measurement projects the qubits into a form that constrains the middle qubit to be $|1\rangle$, but leaves the other 2 qubits indeterminate (since neither qubits A nor C were measured). Moreover, the resulting state must still be properly normalized. Hence, after the measurement, the state of the 3-qubit memory register is:

$$\frac{c_2|010\rangle + c_3|011\rangle + c_6|110\rangle + c_7|111\rangle}{\sqrt{|c_2|^2 + |c_3|^2 + |c_6|^2 + |c_7|^2}}$$

After a partial measurement survive only terms from the initial state of the register that contain the measured qubit state. Bit strings without the measured qubit state are omitted.

Alternative Bases

A basis for an n-qubit quantum memory register is any complete orthonormal set of eigenstates where any n-qubit state can be written as a superposition of states taken from this set only .

Rotations of the single computational basis states are unitary transformations of the computational basis states $|0\rangle$ and $|1\rangle$.

The Bell basis is defined over entangled 2-qubit states.

Typical bases for 2-qubit quantum memory register.

Basis	Eigenstates
θ° Rotated	$\lvert\bar{0}\rangle = \cos\theta\lvert 0\rangle + \sin\theta\lvert 1\rangle$
	$\lvert\bar{1}\rangle = \cos\theta\lvert 0\rangle - \sin\theta\lvert 1\rangle$
Diagonal	$\lvert\nearrow\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$
	$\lvert\searrow\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)$
Chiral	$\lvert\circlearrowright\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle + i\lvert 1\rangle)$
	$\lvert\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle - i\lvert 1\rangle)$
Bell	$\lvert\beta_{00}\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$
	$\lvert\beta_{01}\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$
	$\lvert\beta_{10}\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$
	$\lvert\beta_{11}\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$

How the eigenvectors in the two bases are related?

Rotations in computational basis are represented with a unitary operator: $U = \sum_k |b_k\rangle\langle a_k|$

An operator, *U*, of this form is a unitary matrix that induces the following mapping between the "a"-basis (column vector) and the "b"-basis (column vector):

$$\left.\begin{aligned} |b_1\rangle &= U|a_1\rangle \\ |b_2\rangle &= U|a_2\rangle \\ &\vdots \\ |b_{2^n}\rangle &= U|a_{2^n}\rangle \end{aligned}\right\} \quad \begin{aligned} |b_j\rangle &= U|a_j\rangle \\ \langle b_j| &= \langle a_j|U^\dagger \end{aligned}$$

Thus a given state $|\psi\rangle$ can be written in either the "a"-basis or the "b"-basis as:

$$|\psi\rangle = \sum_i \alpha_i |a_i\rangle = \sum_j \beta_j |b_j\rangle$$

where the amplitudes α_i and β_j are given by the inner products:

$$\alpha_i = \langle a_i|\psi\rangle$$

$$\beta_j = \langle b_j|\psi\rangle$$

β_j amplitudes can be expressed in terms of α_i amplitudes and the unitary matrix elements :

$$\beta_j = \langle b_j|\psi\rangle = \langle b_j| \Big(\underbrace{\sum_i |a_i\rangle\langle a_i|}_{=\boldsymbol{I}}\Big) |\psi\rangle = \sum_i \langle b_j|a_i\rangle\langle a_i|\psi\rangle = \sum_i \langle a_j|U^\dagger|a_i\rangle\langle a_i|\psi\rangle = \sum_i (U^\dagger)_{ji}\alpha_i$$

and $|\psi\rangle_{\text{“}b\text{”-basis}} = \sum_j \beta_j |b_j\rangle = U^\dagger |\psi\rangle_{\text{“}a\text{”-basis}}$

Change of Basis for a State (vector) and for an Operator (observable)

A unitary transformation *U* between the Diagonal and computational bases for a *state vector* is:

$$U = \sum_k |b_k\rangle\langle a_k| = |\nearrow\rangle\langle 0| + |\nwarrow\rangle\langle 1| = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

An *observable* for some property of an *n*-qubit system is represented by a $2^n \times 2^n$ dimensional Hermitian matrix: $\mathcal{O} = \mathcal{O}^\dagger$, and so the eigenvalues $\{\lambda_i\}$ of $\mathcal{O}$ are real numbers for the set of its eigenvectors $\{|\psi_i\rangle\}$:

$$\mathcal{O}|\psi_i\rangle = \lambda_i|\psi_i\rangle$$

Repeated measurements on several preparations of the state $|\psi\rangle$ give an ***average value*** that is always a real number:

$$\langle\mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle$$

where $|\psi\rangle$ and $\mathcal{O}$ should be described with respect to the same basis.

Change of Basis for an Operator $\mathcal{O}$, given initially in the "a"-basis, is expressed by a "similarity matrix transformation" in the "b"-basis :

$$\langle b_k|\mathcal{O}|b_\ell\rangle = \sum_m \sum_n (U^\dagger)_{km}\mathcal{O}_{mn}U_{n\ell}$$

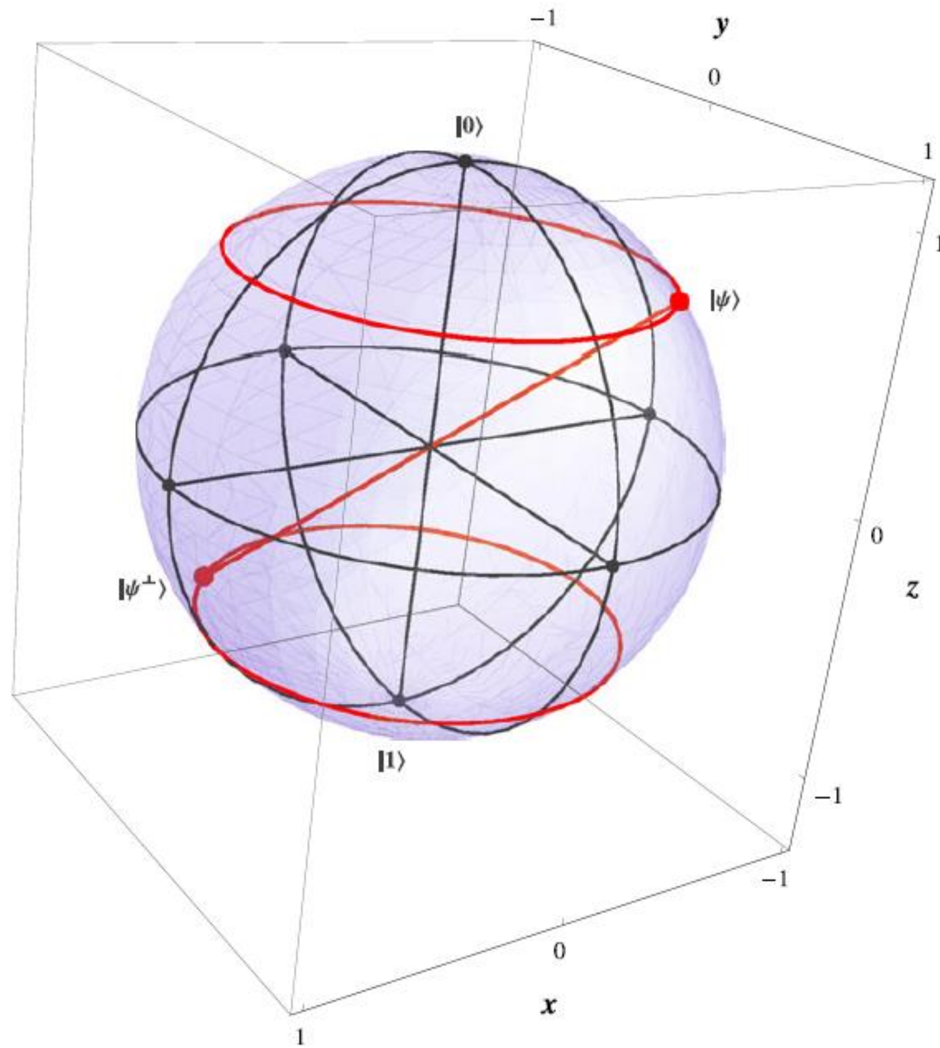
An operator in the "a"-basis transforms into the "b"-basis with a ***similarity matrix transformation***:

$$\mathcal{O}_{\text{“}b\text{”-basis}} = U^\dagger \cdot \mathcal{O}_{\text{“}a\text{”-basis}} \cdot U$$

Observing in an Arbitrary Basis

A given quantum state does not have a unique interpretation: any state—even the state of a quantum memory register—can be pictured as different superposition states over different bases.

Measuring the state of a qubit initially in state $a|0\rangle + b|1\rangle$ along an axis passing through states $|\psi\rangle$ and $|\psi^{\perp}\rangle$ corresponds to measuring the qubit in the basis $\{|\psi\rangle, |\psi^{\perp}\rangle\}$.



Problems with Solutions

Problem #1 Let $\{ |0\rangle, |1\rangle, \ldots, |n-1\rangle \}$ be an orthonormal basis in the Hilbert space $\mathbf{C}^n$. Is $|\psi\rangle = \frac{1}{\sqrt{n}}(\sum_{j=0}^{n-2} |j\rangle \otimes |j+1\rangle + |n-1\rangle \otimes |0\rangle)$ independent of the chosen orthonormal basis?

Solution Consider the special case $\mathbf{R}^2$. Let $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Thus

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Now let

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then

$$|\psi\rangle = \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{\sqrt{2}}\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Thus, $|\psi\rangle$ depends on the chosen basis.

Problems with Solutions

Problem #2 In the product Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$ the *Bell states* are

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle) \quad |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle),$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$ Here, $\{\,|0\rangle, |1\rangle\,\}$ is an arbitrary orthonormal basis in $\mathbf{C}^2$.

Let $|0\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$, $|1\rangle = \begin{pmatrix} -e^{i\phi}\sin\theta \\ \cos\theta \end{pmatrix}$ (i) Find $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, and $|\Psi^-\rangle$ for this basis.
(ii) Consider the special case when $\phi = 0$ and $\theta = 0$.

Solution

(i) We obtain $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{2i\phi}\cos(2\theta) \\ e^{i\phi}\sin(2\theta) \\ e^{i\phi}\sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}$ $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{2i\phi}\sin(2\theta) \\ e^{i\phi}\cos(2\theta) \\ e^{i\phi}\cos(2\theta) \\ \sin(2\theta) \end{pmatrix}$

$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ e^{i\phi} \\ -e^{i\phi} \\ 0 \end{pmatrix}$ (ii) If we choose $\phi = 0$ and $\theta = 0$ which simply means we choose the standard basis for $|0\rangle$ and $|1\rangle$ (i.e., $|0\rangle = (1\ 0)^T$ and $|1\rangle = (0\ 1)^T$), we find

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |\Phi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Problems with Solutions

Problem #3 Let A and B be two $n \times n$ matrices over $\mathbf{C}$. If there exists a non-singular $n \times n$ matrix X such that $A = XBX^{-1}$ then A and B are said to be *similar matrices*.

Show that the spectra (eigenvalues) of two similar matrices are equal.

Solution We have $\det(A - \lambda I_n) = \det(XBX^{-1} - X\lambda I_n X^{-1}) = \det(X(B - \lambda I_n)X^{-1})$

$$= \det(X)\det(B - \lambda I_n)\det(X^{-1}) = \det(B - \lambda I_n)$$

Problem #4

Let $|\psi\rangle = \begin{pmatrix} e^{i\phi}\cos\theta \\ \sin\theta \end{pmatrix}$ where $\phi, \theta \in \mathbf{R}$. (i) Find $\rho := |\psi\rangle\langle\psi|$. (ii) Find $\mathrm{tr}\rho$. (iii) Find ρ^2

Solution (i) Since $\langle\psi| = (e^{-i\phi}\cos\theta, \sin\theta)$ we obtain the 2×2 matrix

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\theta & e^{i\phi}\sin\theta\cos\theta \\ e^{-i\phi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

(ii) Since $\cos^2\theta + \sin^2\theta = 1$ we obtain from (i) $\mathrm{tr}\rho = 1$.

(iii) We have $\rho^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$ since $\langle\psi|\psi\rangle = 1$.

The ***density operator ρ or density matrix*** is a *positive semidefinite* operator on a Hilbert space with unit trace. An operator is positive semidefinite if it *is hermitian* and *none of its real eigenvalues are less than zero*.

Problems with Solutions - Density Operators

The state of a quantum-mechanical system is characterized by a density operator $\boldsymbol{\rho}$ with tr$\boldsymbol{\rho}$ = 1. The expectation value of an observable $\boldsymbol{A}$, determined in an experiment as the average value <$\boldsymbol{A}$> is given by <$\boldsymbol{A}$> = tr($\boldsymbol{A}\,\boldsymbol{\rho}$).

Problem #5 A mixed state is a statistical mixture of pure states, i.e., the state is described by pairs of probabilities and pure states. Given a mixture $\{(p_1, |\psi_1\rangle), \ldots, (p_n, |\psi_n\rangle)\}$ we define its *density matrix* to be the positive hermitian matrix

$$\rho = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j|$$

where the pure states $|\psi_j\rangle$ are normalized (i.e., $\langle\psi_j|\psi_j\rangle = 1$), and $p_j \geq 0$ for $j = 1, 2, \ldots, n$ with

$$\sum_{j=1}^{n} p_j = 1\,.$$

(i) Find the probability that measurement in the orthonormal basis

$$\{|k_1\rangle, \ldots, |k_n\rangle\}$$

will yield $|k_j\rangle$.

(ii) Find the density matrix ρ_U when the mixture is transformed according to the unitary matrix U.

Problems with Solutions - Density Operators

Solution #5 (i) From the probability distribution of states in the mixture we have for the probability $P(k_j)$ of measuring the state $|k_j\rangle$ $(j = 1, 2, \ldots, n)$

$$P(k_j) = \sum_{l=1}^{n} p_l |\langle k_j|\psi_l\rangle|^2 = \sum_{l=1}^{n} p_l \langle k_j|\psi_l\rangle\langle\psi_l|k_j = \langle k_j|\rho|k_j\rangle.$$

(ii) After applying the transform U to the states in the mixture we have the new mixture $\{ (p_1, U|\psi_1\rangle), \ldots, (p_n, U|\psi_n\rangle) \}$, with the density matrix

$$\begin{aligned}\rho_U &= \sum_{j=1}^{n} p_j U|\psi_j\rangle\langle\psi_j|U^* \\ &= U \left(\sum_{j=1}^{n} p_j |\psi_j\rangle\langle\psi_j| \right) U^* \\ &= U\rho U^* .\end{aligned}$$

Problems with Solutions - Density Operators

Problem #6 Let A and B be a pair of qubits and let the density matrix of the pair be ρ_{AB}, which may be pure or mixed. We define the *spin-flipped density matrix* to be

$$\widetilde{\rho}_{AB} := (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$$

where the asterisk denotes complex conjugation in the standard basis

$\{\, |0\rangle \otimes |0\rangle, \quad |0\rangle \otimes |1\rangle, \quad |1\rangle \otimes |0\rangle, \quad |1\rangle \otimes |1\rangle \,\}$ and $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Since both ρ_{AB} and $\widetilde{\rho}_{AB}$ are positive operators, it follows that the product $\rho_{AB}\widetilde{\rho}_{AB}$, though non-hermitian, also has only real and non-negative eigenvalues. Consider the Bell state

$$|\psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

and $\rho := |\psi\rangle\langle\psi|$. Find the eigenvalues of $\rho_{AB}\widetilde{\rho}_{AB}$.

Solution

Since $\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ we have $\widetilde{\rho} = \rho$. Also $\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

Thus $\widetilde{\rho} = \rho$ and $\rho\widetilde{\rho} = \rho$ with eigenvalues 1, 0, 0, 0.