MATH 133 - Formula Sheet

<u>Definition</u>(Norm of a vector)

If $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ the norm of $\mathbf{v}$ is given by $||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

<u>Definition</u>(Dot Product)

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}||\cos\theta$

<u>The Following Are Equivalent</u>

1. $\mathbf{u}$ is orthogonal to $\mathbf{v}$
2. $\mathbf{u} \cdot \mathbf{v} = 0$
3. $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$

<u>Definition</u>(Orthogonal projection)
Let $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$, the projection of $\mathbf{v}$ onto $\mathbf{u}$ is given by

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u}$$

<u>Equation of a line in $\mathbb{R}^2$</u>

If L is a line in $\mathbb{R}^2$ its general equation is given by $\mathbf{ax} + \mathbf{by} = \mathbf{c}$ where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for L.

<u>Equation of a line in $\mathbb{R}^3$</u>
In $\mathbb{R}^3$ the vector equation of a line is given by
$(L) : \mathbf{p} + t\mathbf{d}$

where $\mathbf{p} = (x_0, y_0, z_0)$ is a point on the line and $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is its directional vector.

In parametric form we write $(L) : \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$

Equation of a plane in $\mathbb{R}^3$

Let $\mathbf{p} = (x_0, y_0, z_0)$ be a point in the plane, $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a vector normal to the plane, $\mathbf{u}$ and $\mathbf{v}$ two vectors parallel to the plane (but not parallel to each other) and $\mathbf{x} = (x, y, z)$ any point in the plane then:

- Normal form: $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p})$
- General form: $ax + by + cz = d$
- Vector Form: $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$

Some distances

- distance from a point B to a line (L)
  Get a point A on the line.
  Let $\mathbf{d}$ be the directional vector of the line.
  Denote the vector $\mathbf{AB}$ by $\mathbf{v}$.
  $\text{dist}(B, L) = ||\mathbf{v} - \text{proj}_d\mathbf{v}||$
- distance from a point $B(x_0, y_0)$ to a line $(L) : ax + by = c$ (in $\mathbb{R}^2$)

  $$\text{dist}(B, L) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$$
- distance from a point $B(x_0, y_0, z_0)$ to a plane $(P) : ax + by + cz = d$

  $$\text{dist}(B, P) = \frac{|ax_0 + by_0 + cz_0|}{\sqrt{a^2 + b^2 + c^2}}$$

Theorem (*Number of solution of a system of linear equations*)

A system of linear equations can have only one of the following

- No solution (inconsistent system)
- A unique solution (consistent system)
- A infinte number of solutions (consistent system)

Definition(Elementary Row Operations, **ERO**)
The three elementary row operations are:

1. Interchange two rows.
2. Multiply (or divide) a row by a non-zero constant.
3. Add a multiple of a row to another.

Definition(Reduced Row Echelon Form, **RREF**)
A matrix is in Reduced Row Echelon Form if it satisfies the following 4 conditions

1. All zero rows are at the bottom.
2. The first non-zero entry of every non-zero row is a 1 (leading one).
3. Leading ones go from left to right.
4. All entries above and below any leading one are zero.

If a matrix satisfies only the first 3 conditions above then we say it is in **Row Echelon Form (REF)**.

Definition(Gauss-Jordan elimination process)
This is the process of applying the ERO's to a matrix to get it to RREF.

Definition(Rank of a matrix)
The **rank** of a matrix is the number of non-zero rows in its $RREF$ or REF.

Definition(Linear combination)
A vector $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ if we can find scalars $a_1, a_2, \ldots, a_n$ such that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

Definition(Span, Spanning Set)
Given a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors in $\mathbb{R}^n$:

- Span(S) $=$ the set of all linear combinations of the vectors in S.
- If span(S)$= \mathbb{R}^n$ then we say S is a spanning set for $\mathbb{R}^n$.

Definition(Linear independance)
A set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of vectors in $\mathbb{R}^n$ is said to be linearly independant if the only solution to the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$. Otherwise the vectors are called linearly dependant (which also means that at least one of them can be written as a linear combination of the others).

Definition(Symmetric matrix)
A square matrix is symmetric if $A = A^T$.

Definition(Inverse of a Square Matrix)
Given a square matrix A its inverse (if it exists) is the matrix denoted by A^{-1} such that $AA^{-1} = A^{-1}A = I$.

If the matrix is a 2×2 matrix we use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided that the determinant of A, $\det(A) = ad - bc \neq 0$.
For a matrix of higher dimensions the process looks like this:

$$[A \mid I] \rightarrow \text{Gauss Jordan Process} \rightarrow \left[I \mid A^{-1}\right]$$

If the matrix is not invertible (*i.e.* does not have an inverse) we will not get the identity on the left side after applying the Gauss-Jordan process.

Definition(Elementary Matrix)
An elementary matrix is a matrix that can be obtained by applying one Elementary Row Operation to the identity matrix.

Definition(Row Space, Column Space, Null Space)
Let A be an $m \times n$ matrix,

- The row space of A = span(Rows of A).
- The Column space of A = span(Columns of A).
- The Null space is the subspace of $\mathbb{R}^n$ spanned by the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Definition(Basis)
A basis of a subspace S of $\mathbb{R}^n$ is a set of vectors that span S and are linearly independant.

Definition(Rank)
The rank of a matrix A (denoted by rank(A))is the dimension of its row space (or column space since they're equal)

Definition(Nullity)
The nullity of a matrix A (denoted by nullity(A)), is the dimension of its Null space.

Theorem (*The Rank Theorem*)
For any $A_{m \times n}$,

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Definition(Linear Transformation)
A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a linear transformation if it satisfies

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(k\mathbf{u}) = kT(\mathbf{u})$

We usually check if T is a linear transformation by checking that

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$

for c_1, c_2 scalars and $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{R}^n$.

Definition(Minor)
Given $A_{n\times n}$, the minor of entry ij is denoted by A_{ij} and is the determinant of the matrix we get from A by removing row i and column j.

Definition(Cofactor)

$$C_{ij} = (-1)^{i+j} A_{ij}$$

Definition(Determinant of an $n \times n$ matrix)
Given an $n \times n$ matrix A ($n \geqslant 2$)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

by expanding along the i^{th} row.

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

by expanding along the j^{th} column.

Properties of the determinant function
Given an $n \times n$ matrix A

- If A has a zero row or zero column then $\det(A) = 0$.
- If we get matrix B by interchanging two rows of A then $\det(B) = -\det(A)$.
- If we get matrix B by multipying one row of A by $k \neq 0$ then $\det(B) = k\det(A)$.
- If we get matrix B by adding a multiple of a row to another of matrix A then $\det(B) = \det(A)$.
- $\det(kA) = k^n \det(A)$.
- $\det(A^T) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Definition(Eigenvalue, Eigenvector, Eigenspace)
Given $A_{n\times n}$ a scalar λ is an eigenvalue of A if there is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.
The eigenvalues of A are the roots of the characteristic polynomial given by $\det(A - \lambda I)$; (we solve $\det(A - \lambda I) = 0$).
In this case $\mathbf{x}$ is called an eigenvector or A corresponding to λ.
The collection of all eigenvectors corresponding to λ along with the zero vector form the eigenspace of λ denoted by E_λ.

Definition(Similar Matrices)
Given A and B two $n \times n$ matrices. A is said to be similar to B (written $A \sim B$) if there is an invertible matrix P such that $P^{-1}AP = B$.

Definition(Diagonalizable matrix)
An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D that is similar to A. *i.e.* If there is a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

Theorem (*when is a matrix diagonalizable?*)
An $n \times n$ matrix A is diagonalizable if one of the following is true

- A has n distinct eigenvalues.
- For each eigenvalue the geometric multiplicity is equal to the algebraic multiplicity.

Definition(Orthogonal set)
A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set if any two vectors in the set are orthogonal. (*i.e.* $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i, j = 1, \ldots n$).

Definition(Orthogonal basis)
An orthogonal basis is a basis that is also an orthogonal set.

Definition(Orthogonal matrix)
An $m \times n$ matrix Q is called orthogonal if $Q^T Q = I_n$.
(The columns of Q form an orthonormal set)

Theorem (*Important property about Orthogonal matrices*)
If Q is a square orthogonal matrix then $Q^T = Q^{-1}$.

Definition(Orthogonal complement)
Let W be a subspace of $\mathbb{R}^n$. We say that a vector $\mathbf{v}$ in $\mathbb{R}^n$ is orthogonal to W if $\mathbf{v}$ is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the Orthogonal complement of W and denoted by $\mathbf{W}^\perp$.

Theorem (*Important theorem to find* $\mathbf{W}^\perp$)
If A is an $m \times n$ matrix then then

$$(row(A))^\perp = null(A) \quad \text{and} \quad (col(A))^\perp = null(A^T)$$

Definition(Orthogonal projection of $\mathbf{v}$ onto W)
Let W be a subspace of $\mathbb{R}^n$ and let $\{u_1, u_2, \ldots, u_k\}$ be an orthogonal basis for W. For any vector $\mathbf{v}$ in $\mathbb{R}^n$, the orthogonal projection of $\mathbf{v}$ onto W is given by

$$\text{proj}_W \mathbf{v} = \left(\frac{\mathbf{u}_1 \cdot \mathbf{v}}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 + \cdots + \left(\frac{\mathbf{u}_k \cdot \mathbf{v}}{\mathbf{u}_k \cdot \mathbf{u}_k}\right)\mathbf{u}_k$$

Definition(The Gram-Schmidt process)
The Gram-Schmidt process is the process we use to transform a basis into an orthogonal basis. It works as follows:

Given $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ a basis for a subspace W of $\mathbb{R}^n$

$$
\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{x}_2 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \mathrm{proj}_{\mathbf{v}_1}\mathbf{x}_3 - \mathrm{proj}_{\mathbf{v}_2}\mathbf{x}_3 \\
&\vdots \\
\mathbf{v}_k &= \mathbf{x}_k - \mathrm{proj}_{\mathbf{v}_1}\mathbf{x}_k - \mathrm{proj}_{\mathbf{v}_2}\mathbf{x}_k - \cdots - \mathrm{proj}_{\mathbf{v}_k}\mathbf{x}_k
\end{aligned}
$$

Finally we have $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ an orthogonal basis for W.