UNIVERSITY OF PATRAS

DEPARTMENT OF BUSINESS ADMINISTRATION

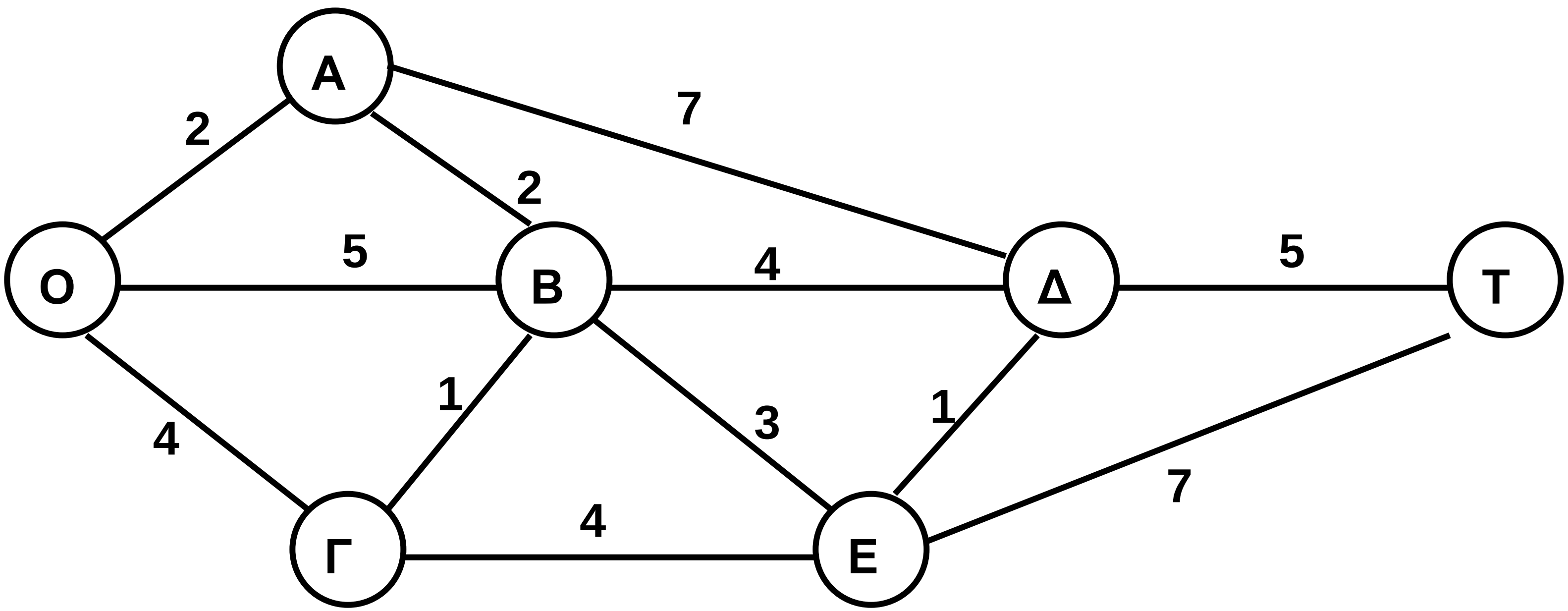
FURTHER OPERATIONAL RESEARCH TECHNIQUES

Lecture 2: Other Problems in Graphs

Patras 2022

The Minimum Spanning Tree Problem (MST)

- Assume the national park of the previous lecture



- If all kiosks must be connected by phone lines, what is the minimum total length of lines required?

Properties of the Solution

- **The solution to the MST is a tree, i.e.**
  - **It has n-1 edges**
  - **It has no cycles (circuits)**
- **The total length of this tree is minimal**
- **The final solution is not affected by the choice of starting node**

Algorithm

1 Choose a node at random and connect it to its nearest neighbor

2 Repeat until the end

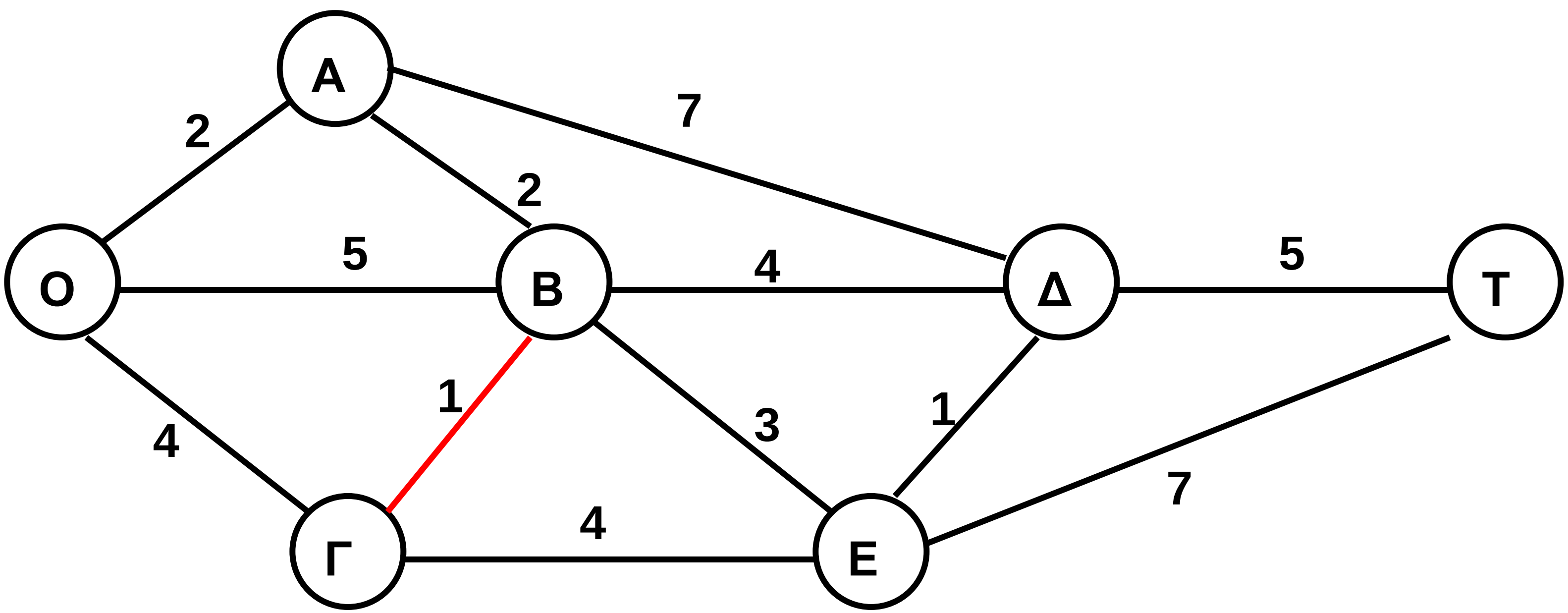
  2.1 Find the non-connected node which is closest to one of the already connected nodes

  2.2 Connect these two nodes

(Optimal Solution of the Example)

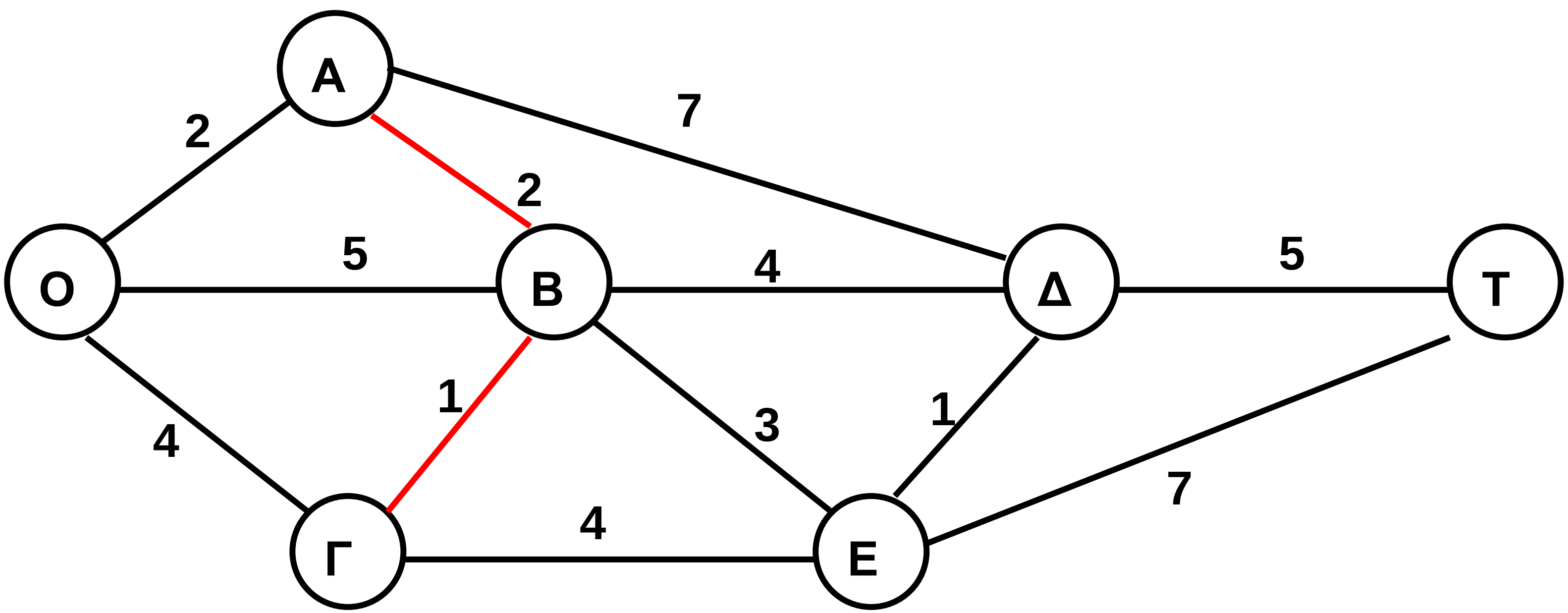
- Lets start (arbitrarily) with node B

- 1st iteration: Connect node Γ



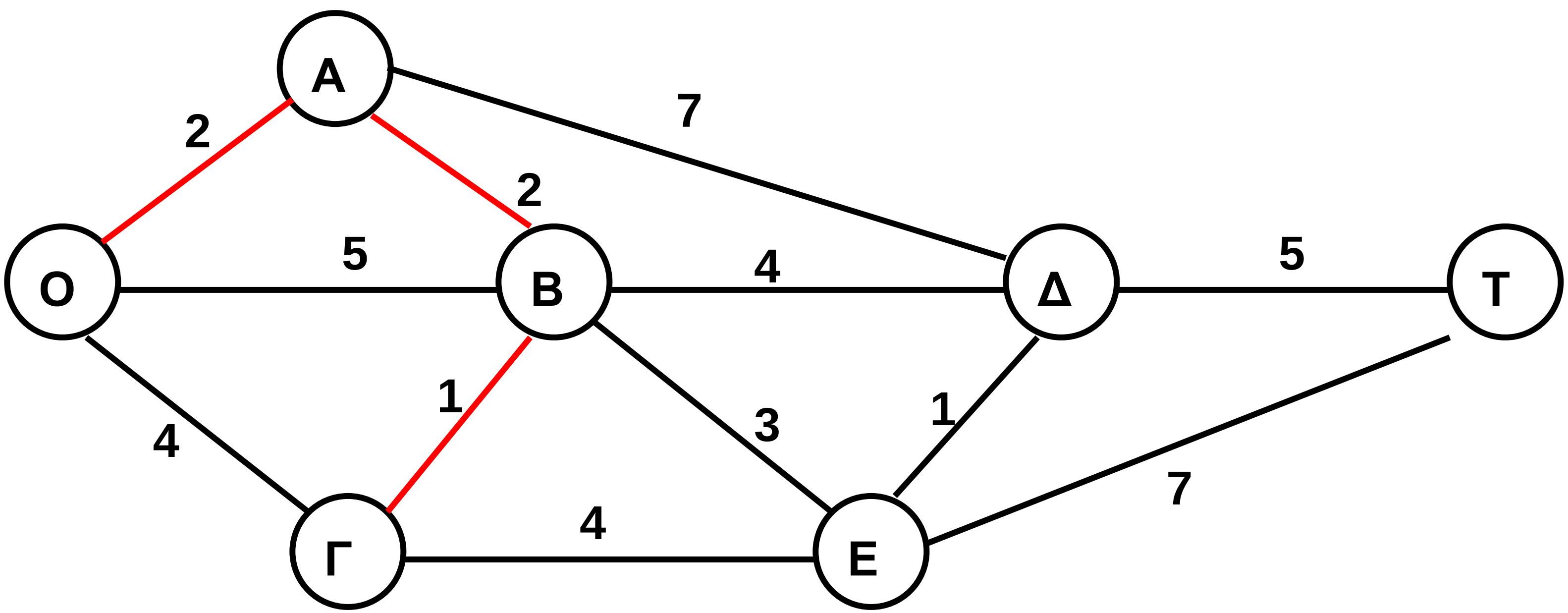
(Optimal Solution of the Example)

- 2nd iteration: Connect node A



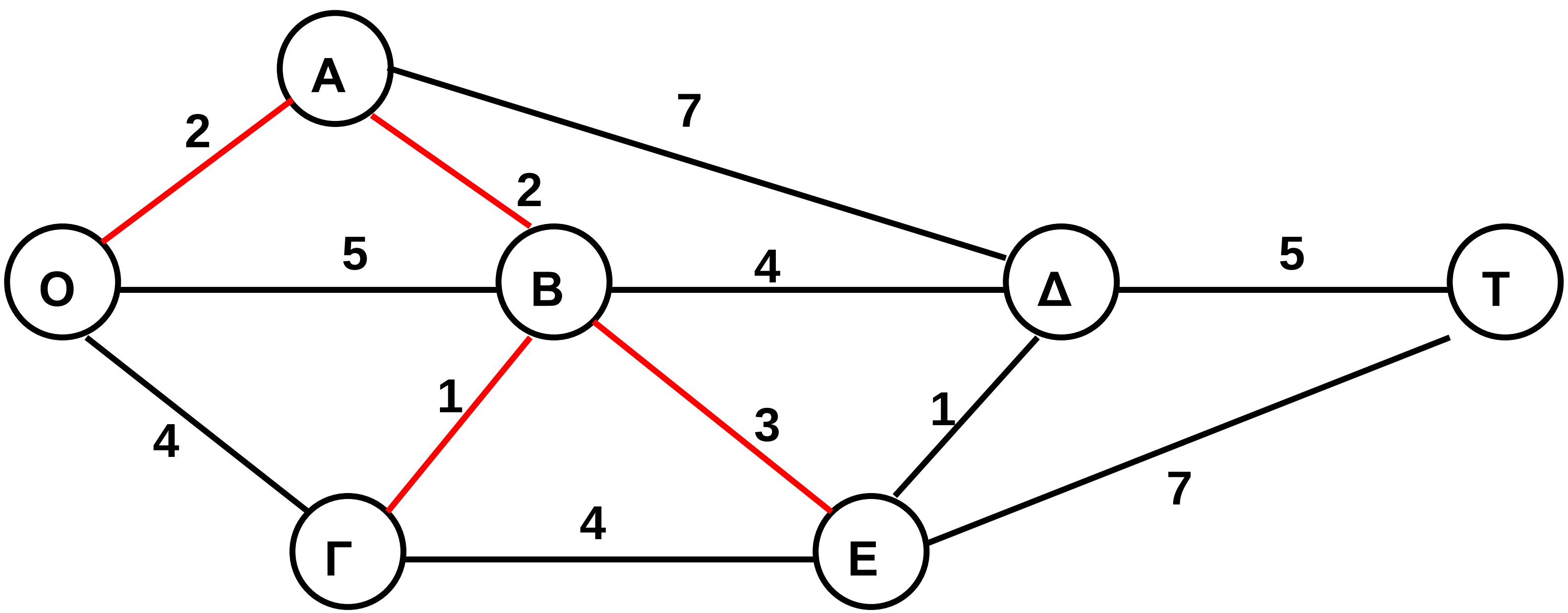
(Optimal Solution of the Example)

- 3^d iteration: Connect node O



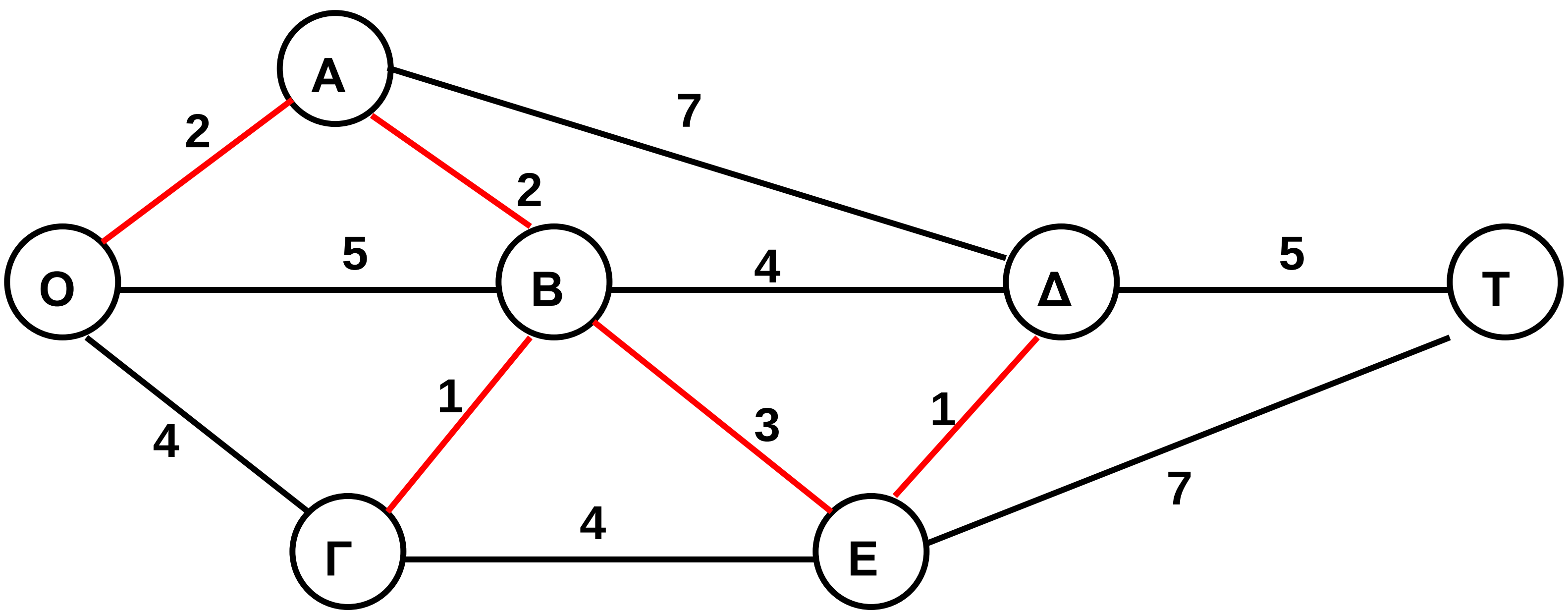
(Optimal Solution of the Example)

- 4^{th} iteration: Connect node E



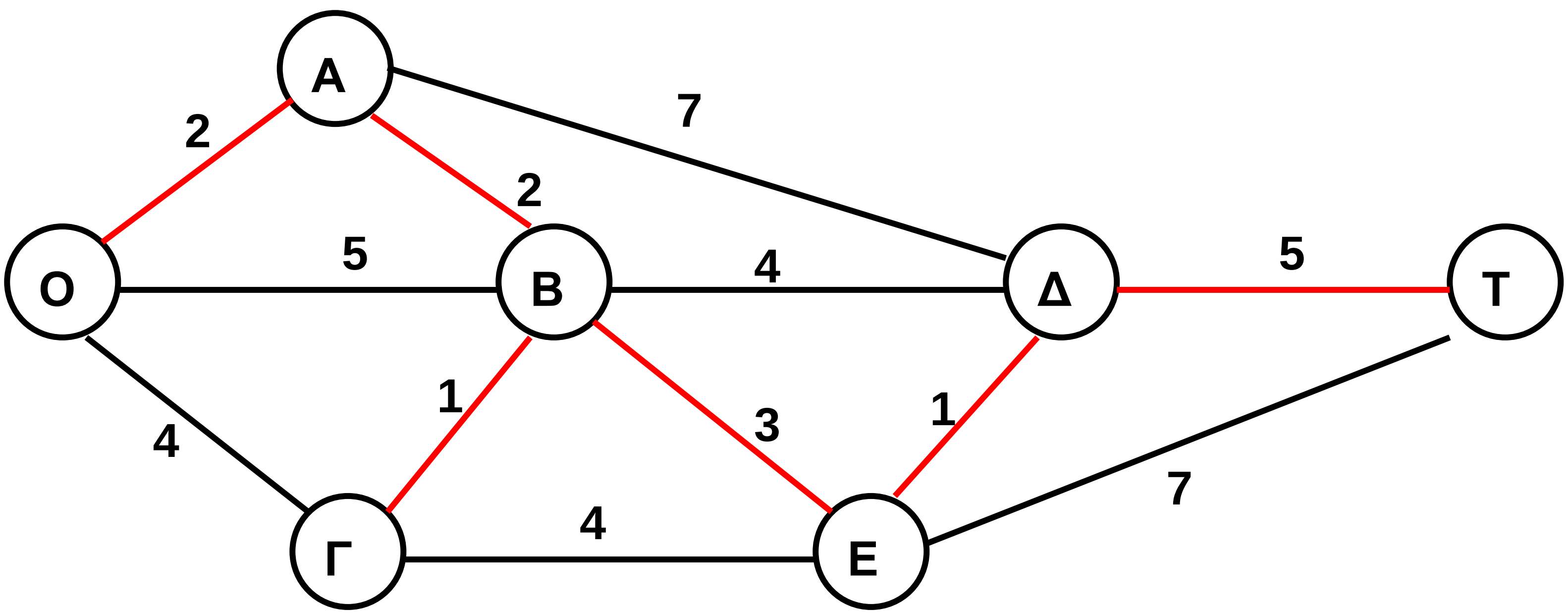
(Optimal Solution of the Example)

- 5^{th} iteration: Connect node Δ



(Optimal Solution of the Example)

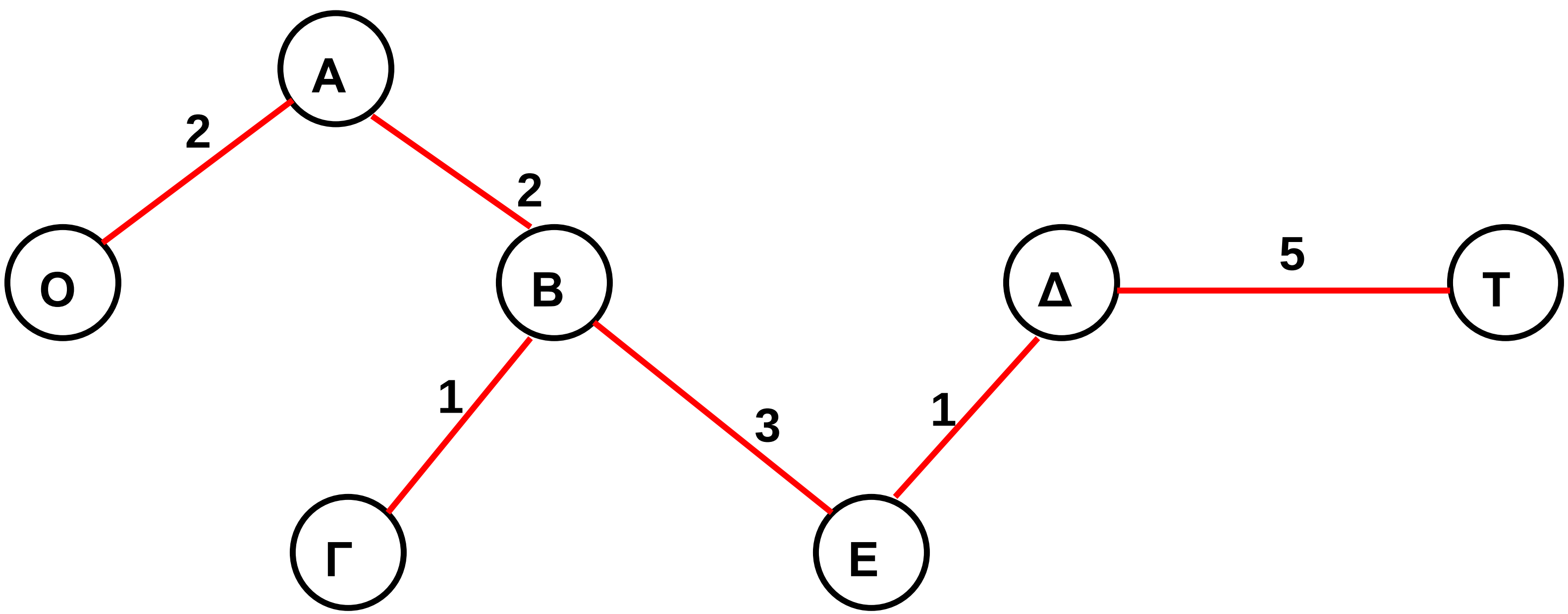
- 6th iteration: Connect node T



(Optimal Solution of the Example)

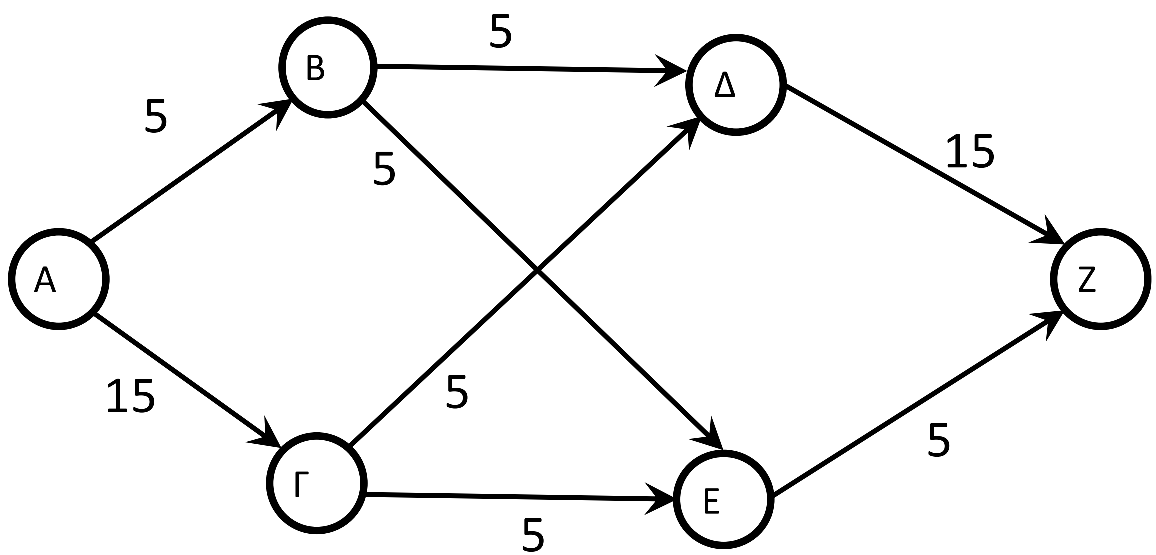
- Final Solution: minimum total length 2+2+1+3+1+5=14

- Tree with 7 nodes and 6 edges



The Maximum Flow Problem- Example

- Assume the electricity distribution network of an area
- Node A denotes the power generating plant and node Z a concert hall. The other nodes denote intermediary distribution nodes whereas the edges denote power cables.
- Every edge has a certain capacity (in kWh).



(What is the maximal energy that can be distributed from the plant (A=so) to the concert hall (Z=si)?

The Maximum Flow Problem- Definition

- **Let a directed graph G=(V,E)**
- **Let *so (source)* the origin and *si (sink)* the terminal node**
- **Let $u_{ij} > 0$ the capacity of every edge $(i, j) \epsilon E$**
- **Problem: for every edge $(i, j) \epsilon E$ find flow x_{ij} such that**
  - **At every node (except *so* and *si*) incoming flow must be equal to outgoing flow (maintenance condition)**
  - **For every edge $(i, j) \epsilon E$ we must have $x_{ij} \leq u_{ij}$**
  - **The total incoming flow into the sink (*si*) must be maximized**

Important Concept: Augmenting Path

- **Solving the problem relies on our ability to find routes along which we can increase the flow**

- **<u>Augmenting Path</u>: A path so=v_1,v_2,…,v_r=si such that:**
  - **We can increase the flow when moving in the same direction as the edge**
  - **We can decrease existing flow when moving in the opposite direction of the edge**

- **(It doesn't matter which order we visit the edges)**

The Maximum Flow Problem (continued)

- **Basic idea: Augmenting path**
  - **(Non-directed) path *so*=v_1,v_2,…,v_r=*si* such that for every edge $(i, j)\,\epsilon\,E$ it must be**

$$u_{ij} - x_{ij} > 0 \text{ or } x_{ji} > 0$$

There exists remaining capacity when moving in the direction of the edge (for $u_{ij} - x_{ij} > 0$)

There exists non-zero flow when moving against the direction of the edge (for $x_{ji} > 0$)

- **If such a path exists, let δ the maximum increase in flow that can be achieved**

Algorithm (Ford-Fulkerson)

1 Find a feasible flow (x_{ij}) with value z ($i,j \in V$)

2 Find an augmenting path P.

If no such path exists, then the solution is optimal.

Otherwise, go to Step 3.

3 Let δ the maximum possible increase of the flow

Increase the flow along the path as follows:

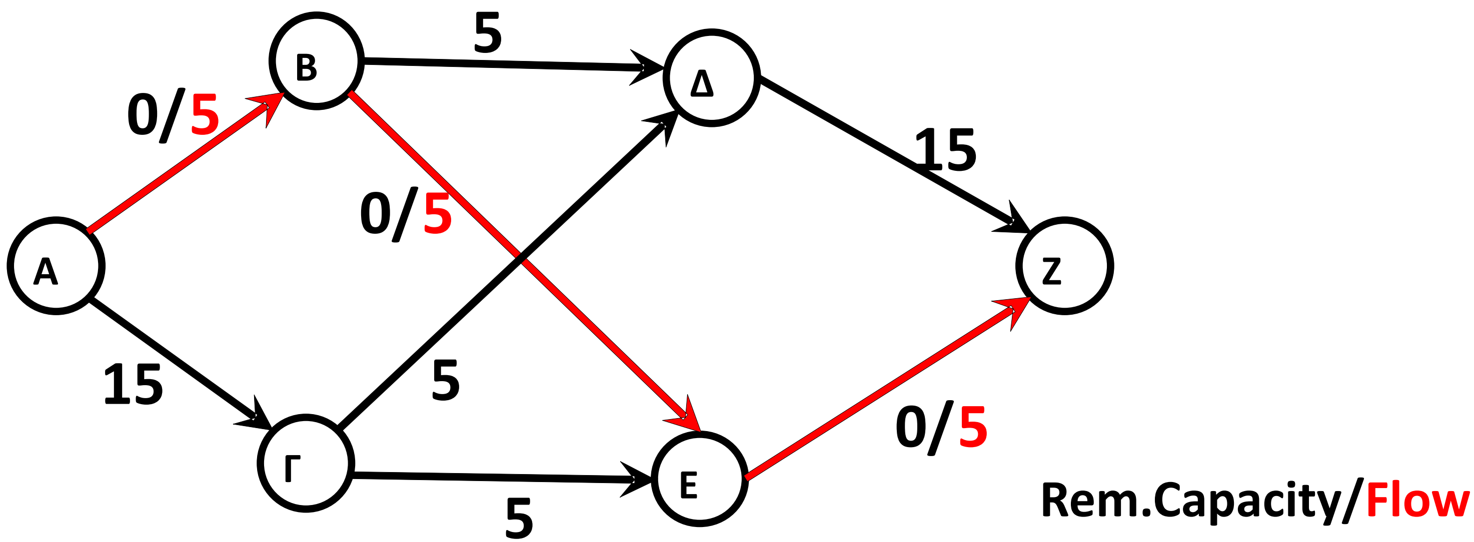
$$x'_{ij} = \begin{cases} x_{ij} + \delta, \text{ if } (\mathrm{i}, \mathrm{j}) \in \mathrm{P} \\ x_{ij} - \delta, \text{ if } (\mathrm{j}, \mathrm{i}) \in \mathrm{P} \end{cases}$$

Then (x'_{ij}) is a feasible flow with value $z' = z + \delta$ ($i,j \in V$)

4 Return to Step 2

Return to the Example

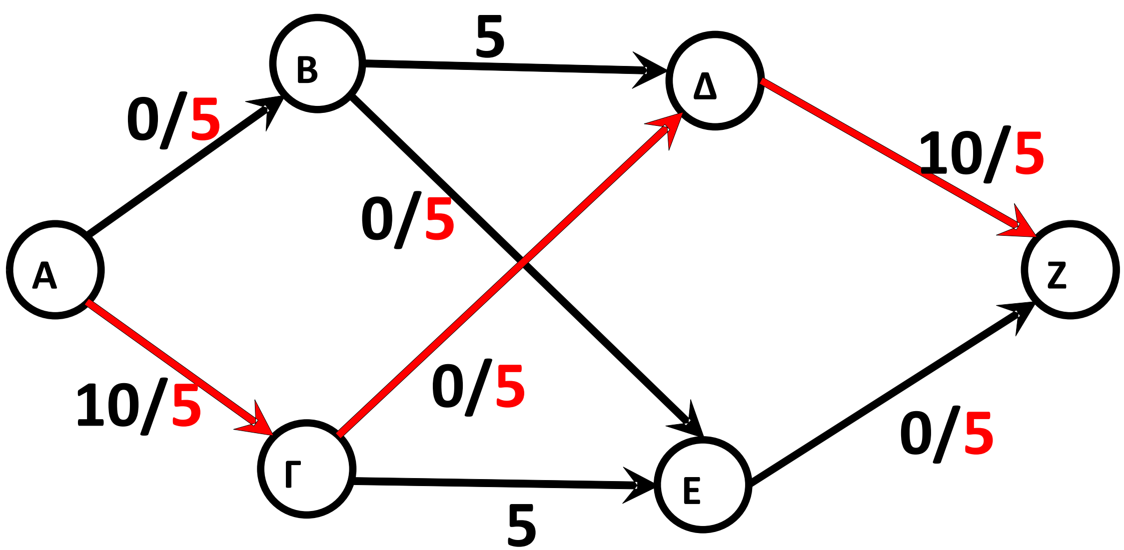
- Augmenting Parth AB-BE-EZ with flow 5



- Total flow: 5

Return to the Example / 2

- Augmenting Path ΑΓ-ΓΔ-ΔΖ with flow 5

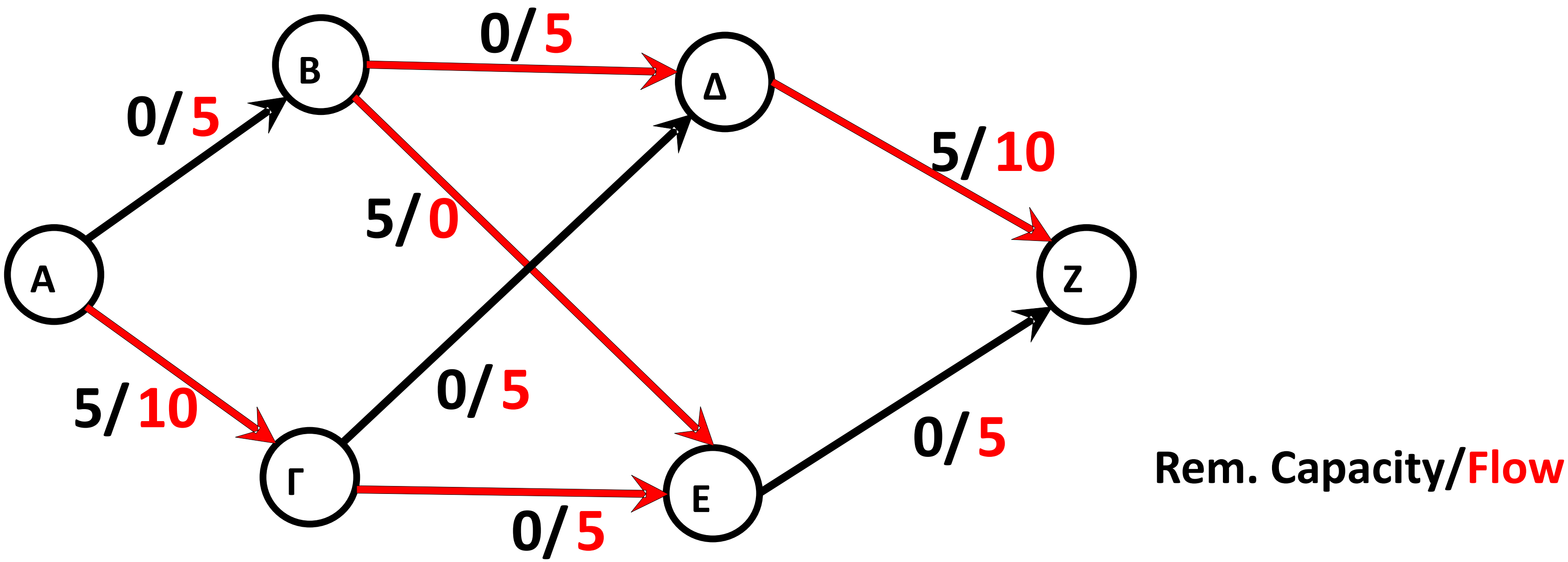


Rem. Capacity/Flow

- Total flow: 5+5=10

Back to the Example/ 3

- Augmenting Path ΑΓ-ΓΕ-ΕΒ-ΒΔ-ΔΖ with flow 5

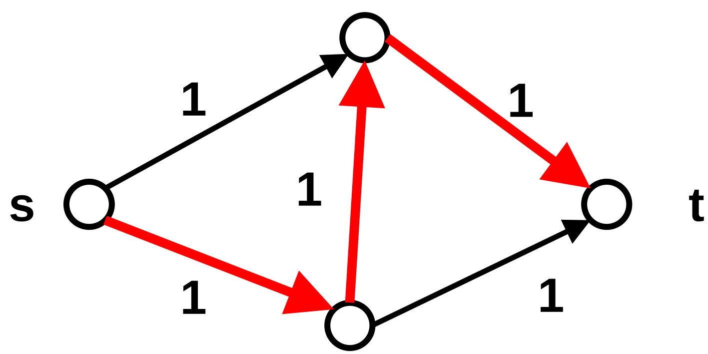


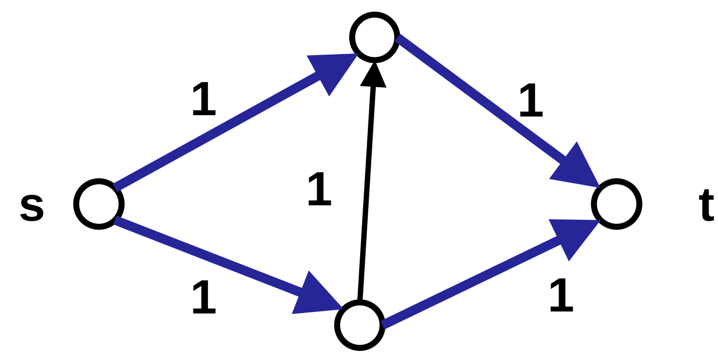
- We can move along EB (against the direction of the edge) because we have already sent positive flow along BE

- Total flow: 5+5+5=15 (Maximal!)

Ford-Fulkerson Algorithm /observation

- Moving against the direction of an edge, we basically reduce the flow along this edge
- This reduction allows us to change previous flows
- Example:



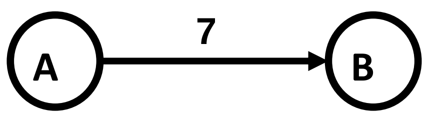


- Total flow=1

Total flow =2

The residual network

- Technique to implement the Ford-Fulkerson algorithm
- Shows the remaining capacity for each edge
- This is the maximum flow we can send along that edge
- Example: let an edge (A, B) with capacity 7. This edge is represented as follows:

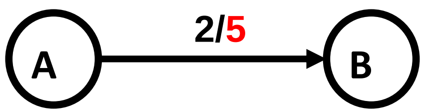


Actual Network

Residual Network

The residual network/2

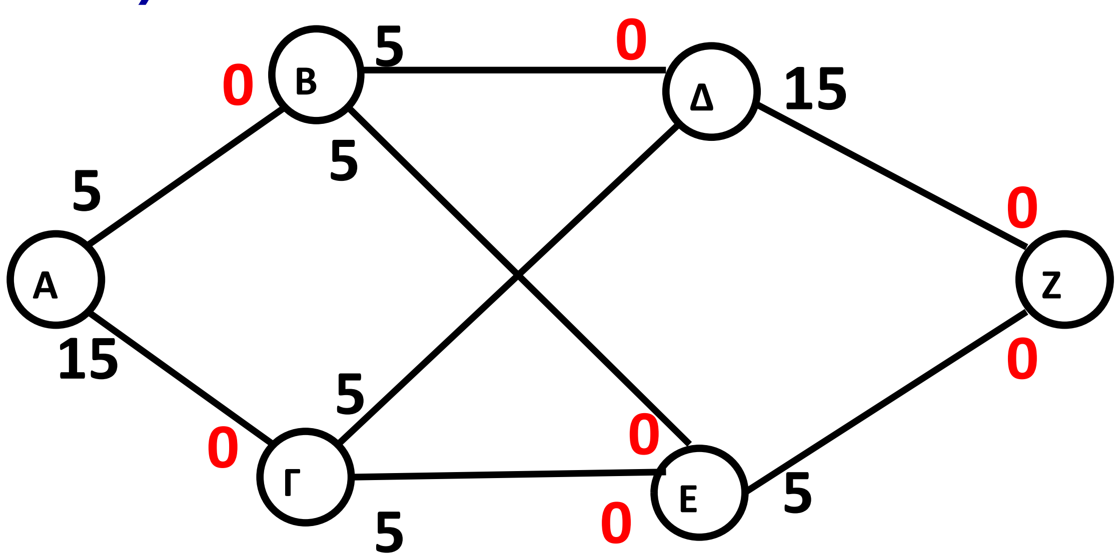
- Example: if we send a flow of 5 units along edge (A, B), this flow is represented as follows:



Actual Network

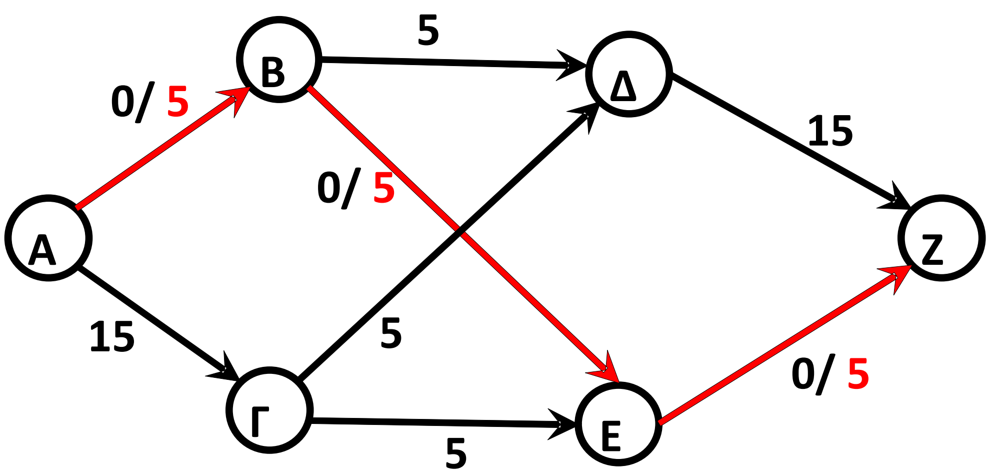
Residual Network

- The residual network of the initial example (electricity distribution) is:

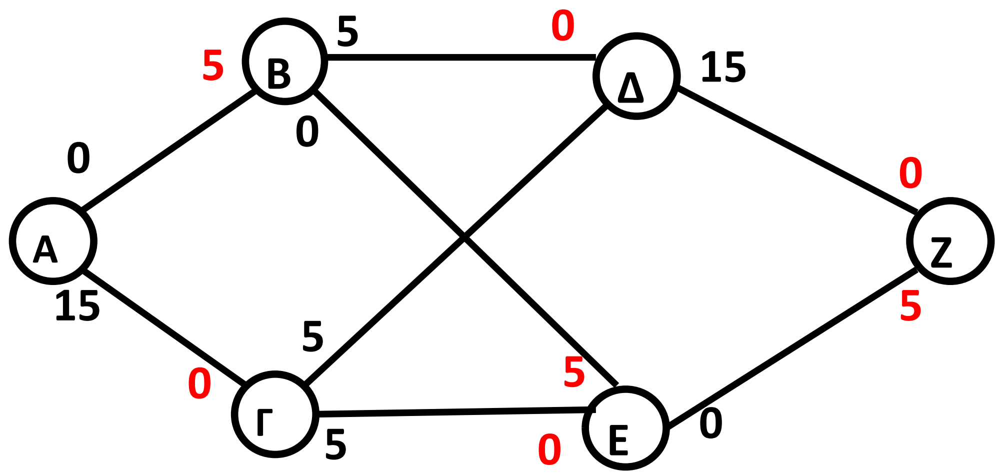


The residual network/3

- After the first iterations (send a flow of 5 units along the path) the network is as follows:



Actual Network



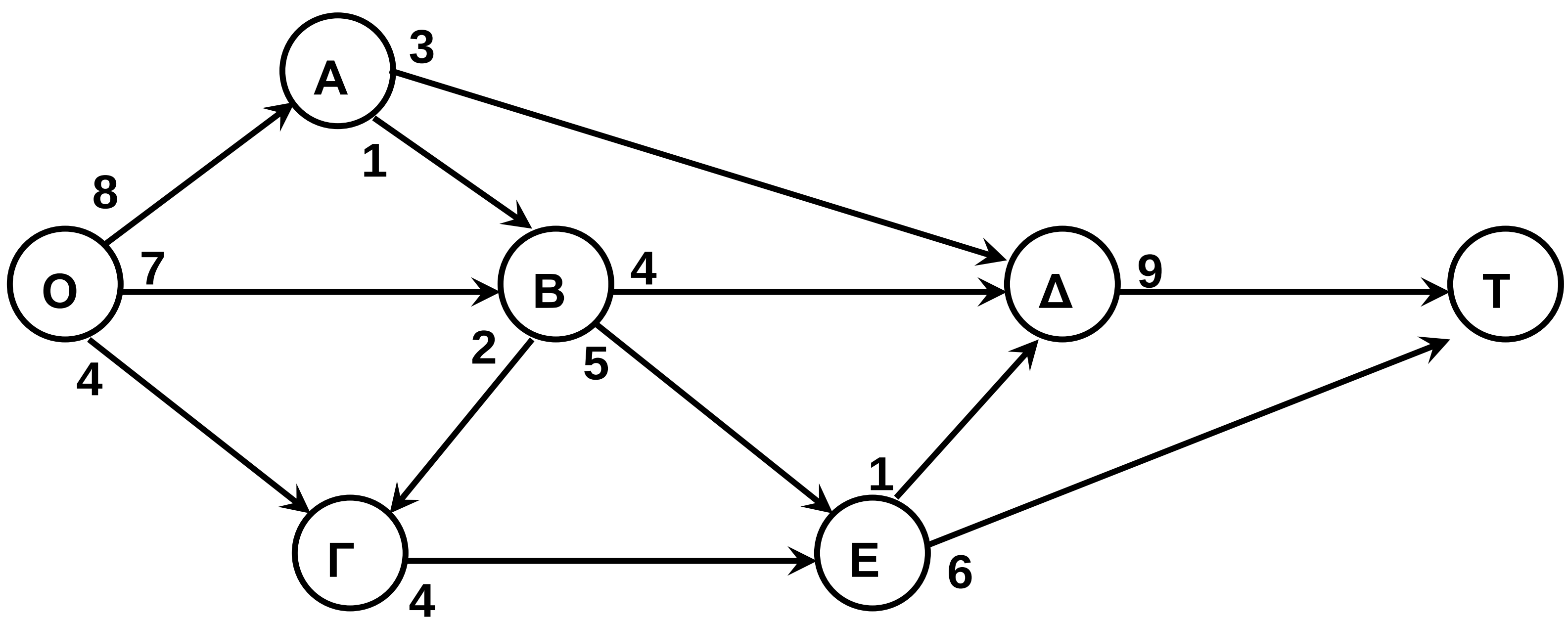
Residual Network

The Maximum Flow Problem /Example 2

- **Assume the national park again**

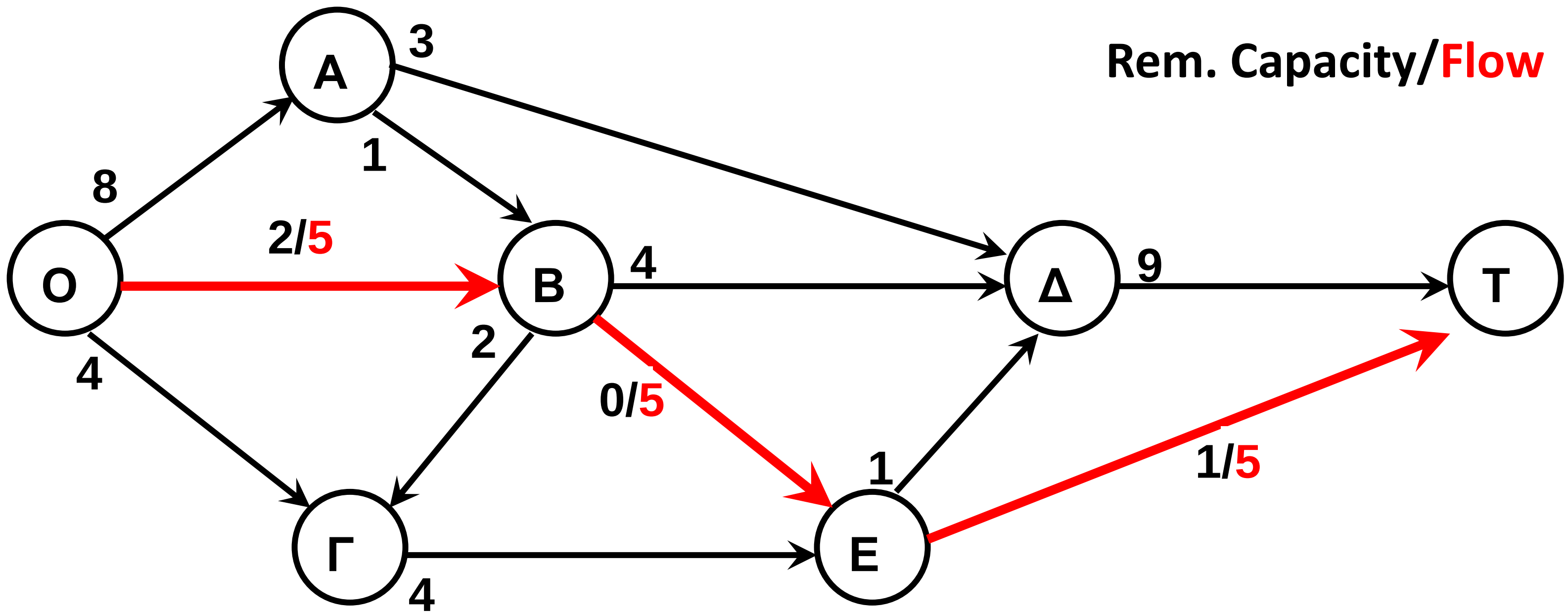
- **Each road has a certain capacity i.e. it may accept a limited number of cars per unit time (see graph)**

- **What is the maximum number of cars that can travel from the entrance (O=*so*) to the exit (T=*si*) of the park per unit time?**



Solution / 1

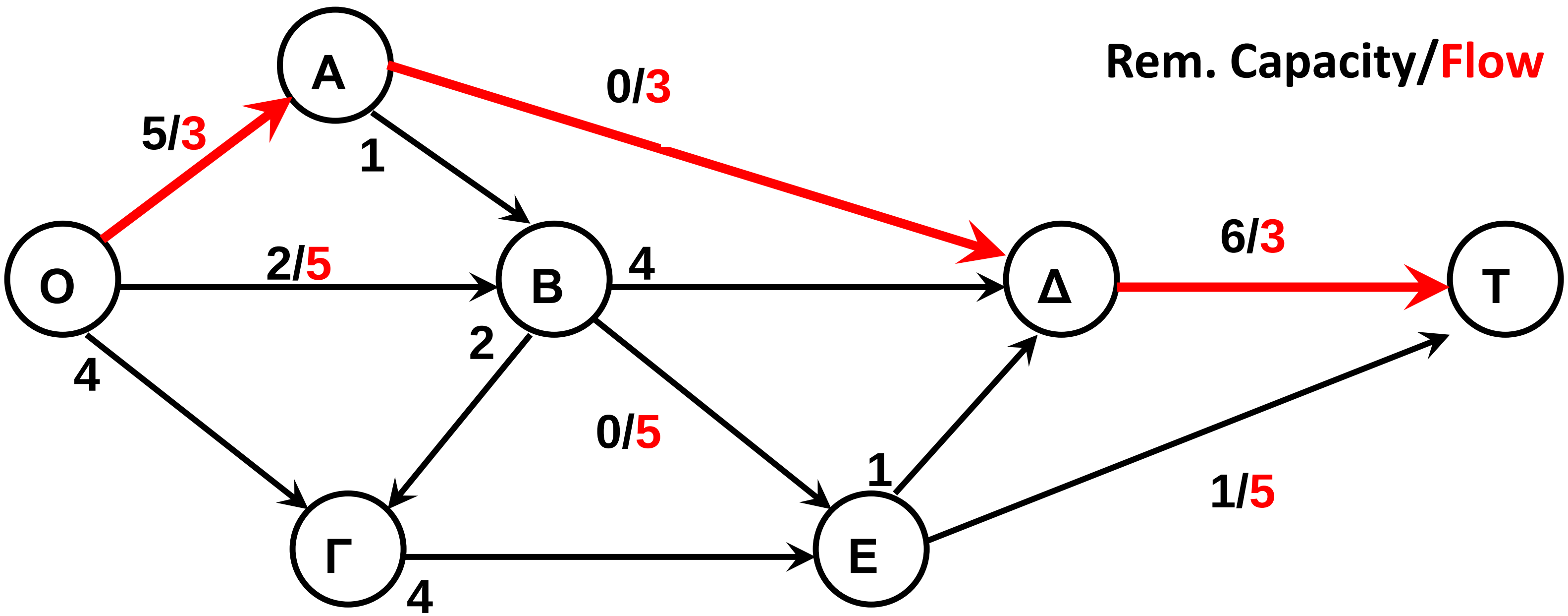
- **Augmented Path OB-BE-ET with flow 5**



- **Total flow 5**

Solution / 2

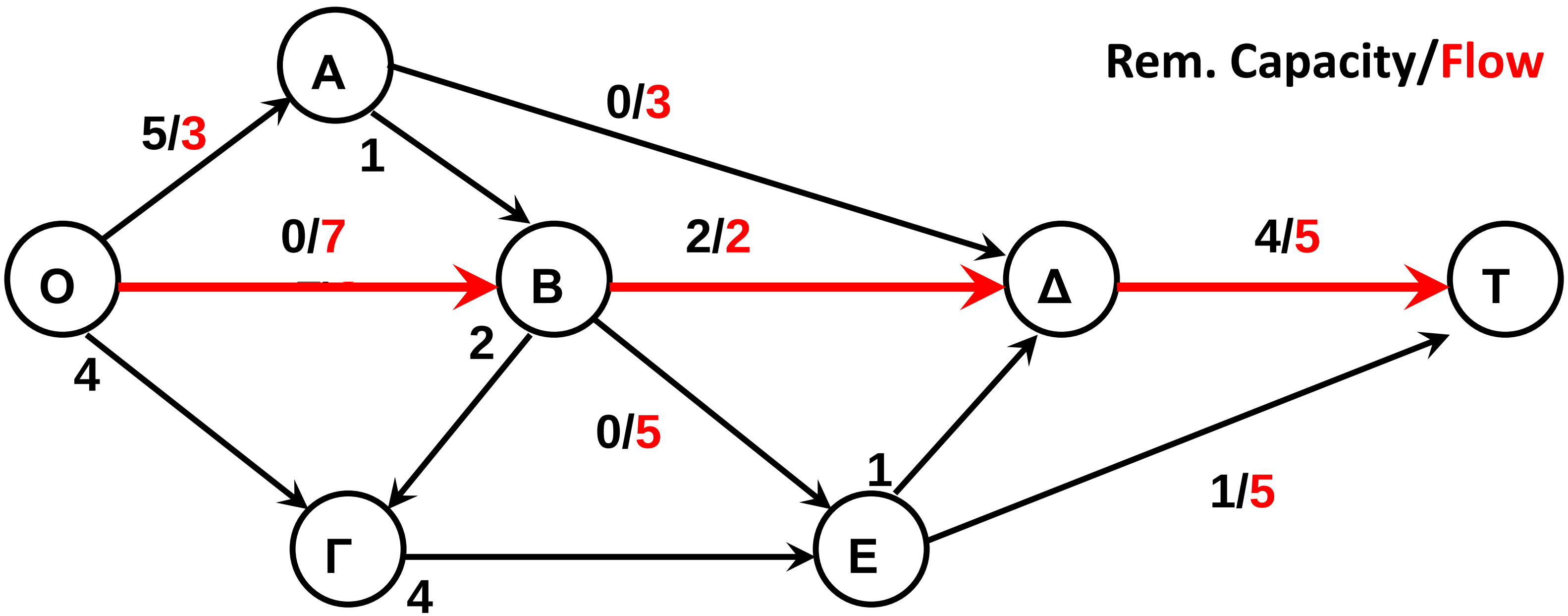
- **Augmented Path OA-AΔ-ΔT with flow 3**



- **Total flow 5+3=8**

Solution / 3

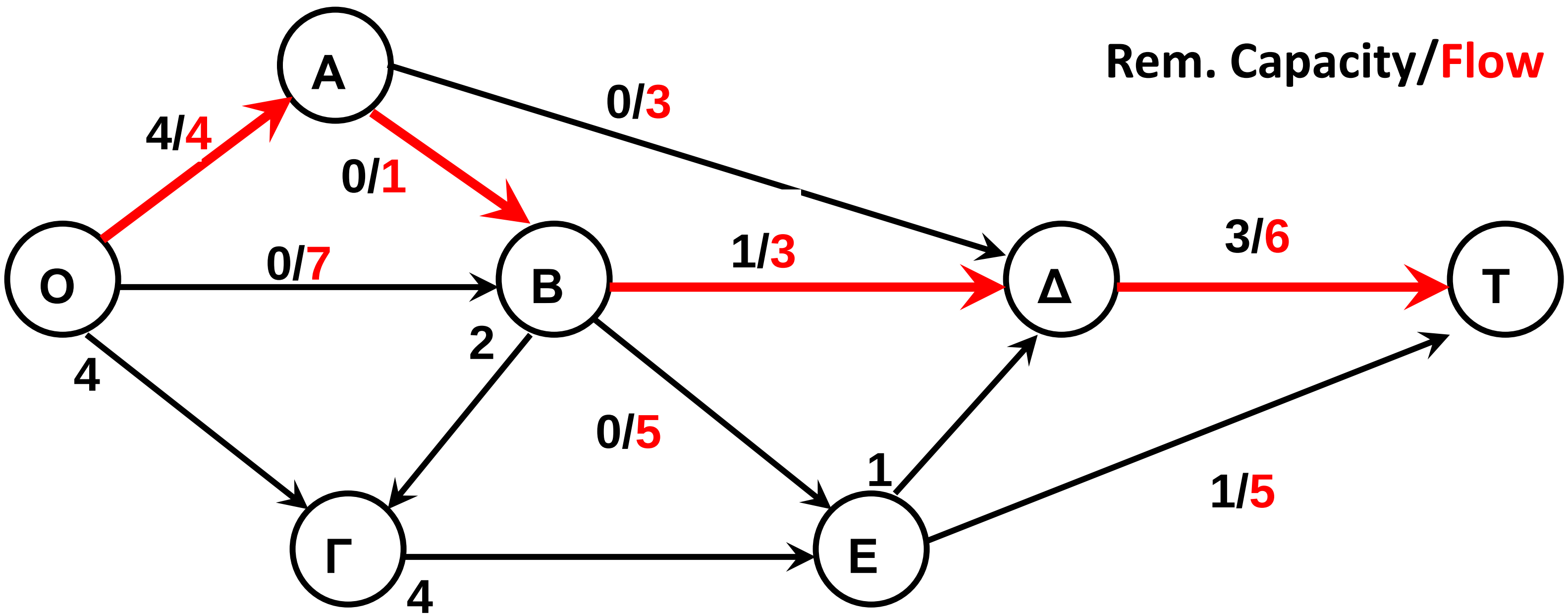
- Augmented Path OB-BΔ-ΔT with flow 2



- Total flow 5+3+2=10

Solution / 4

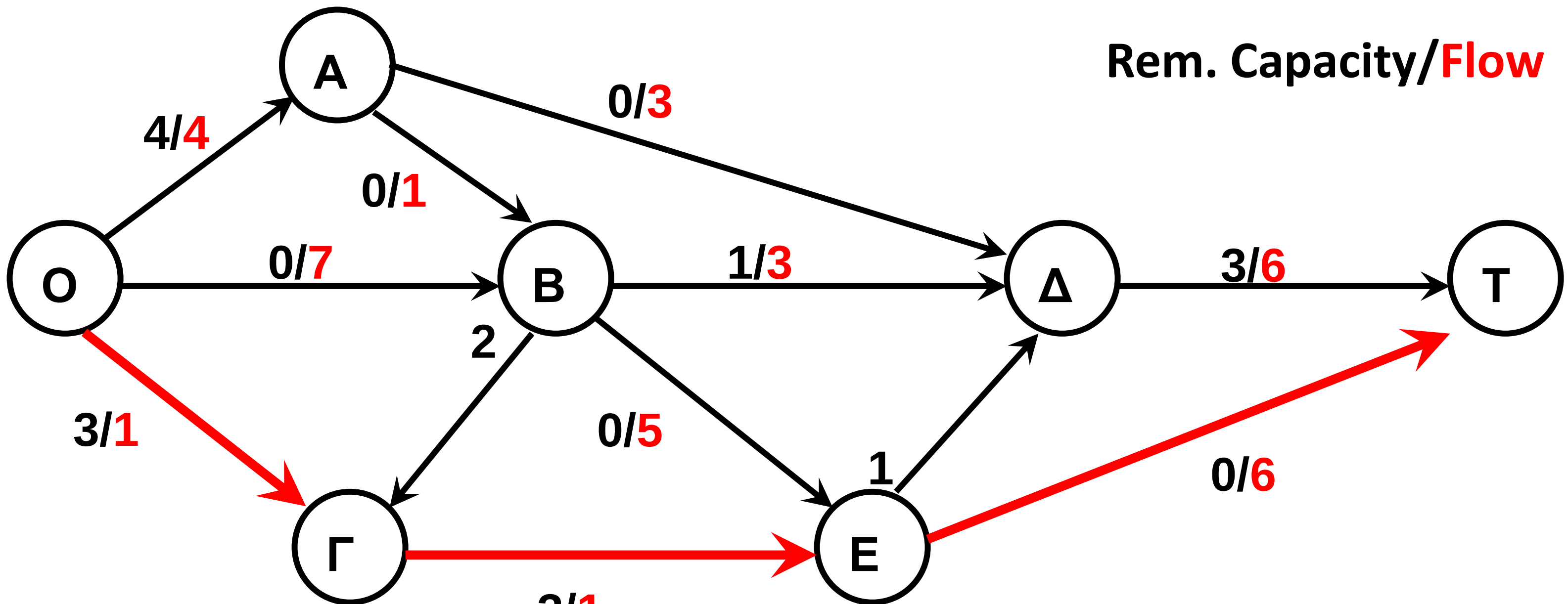
- **Augmented Path OA-AB-BΔ-ΔT with flow 1**



- **Total flow 5+3+2+1=11**

Solution / 5

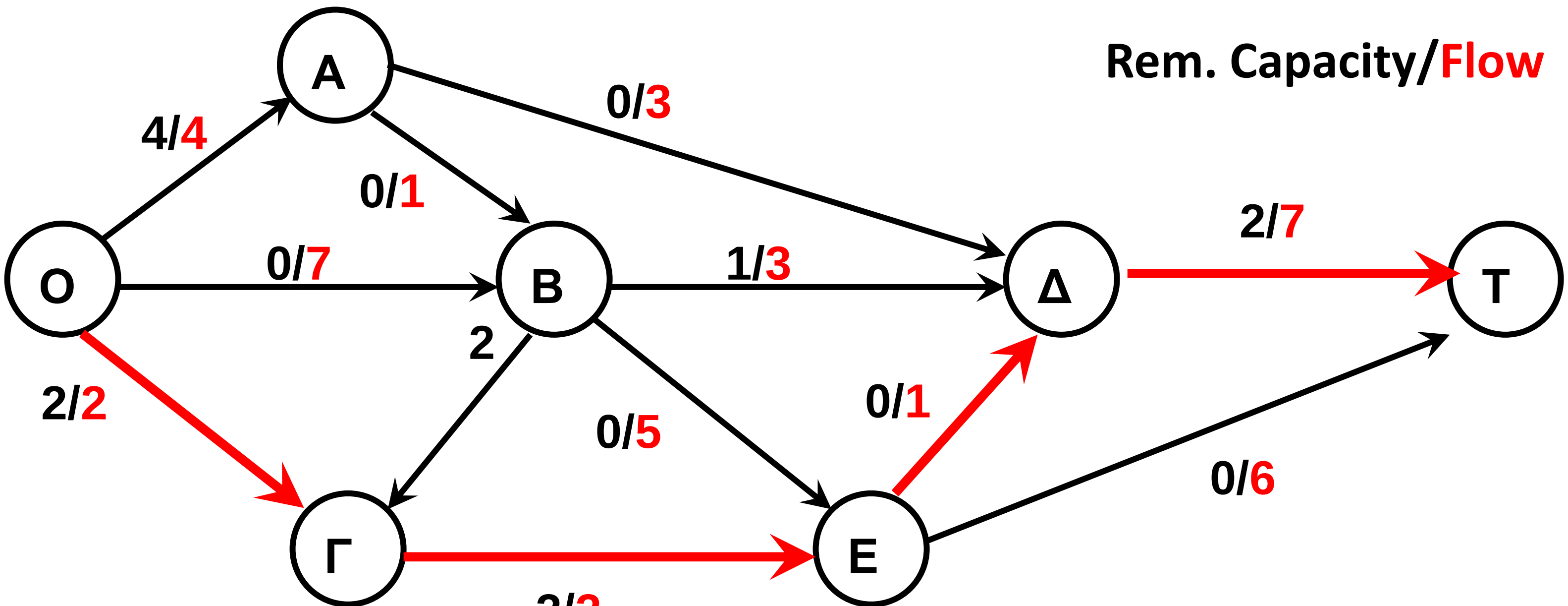
- **Augmented Path ΟΓ-ΓΕ-ΕΤ with flow 1**



- **Total flow 5+3+2+1+1=12**

Solution / 6

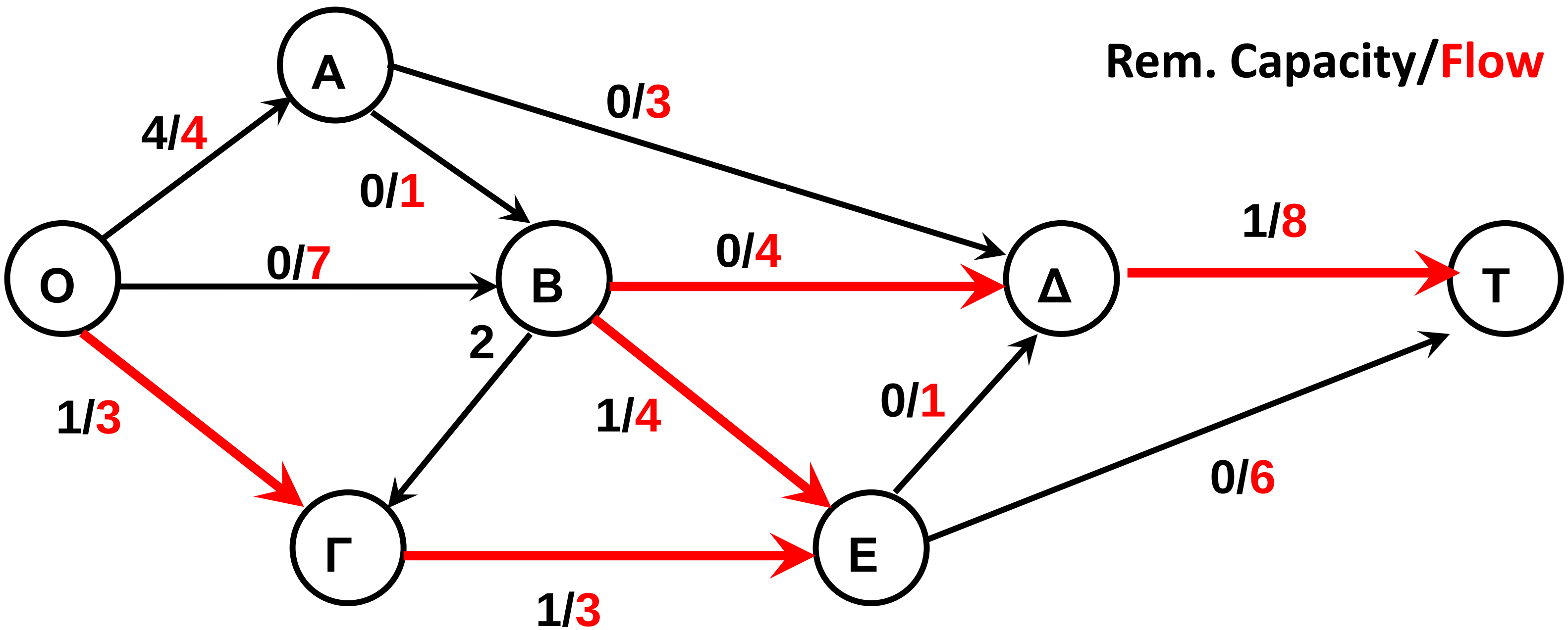
- **Augmented Path ΟΓ-ΓΕ-ΕΔ-ΔΤ with flow 1**



- **Total flow 5+3+2+1+1+1=13**

Solution / 7

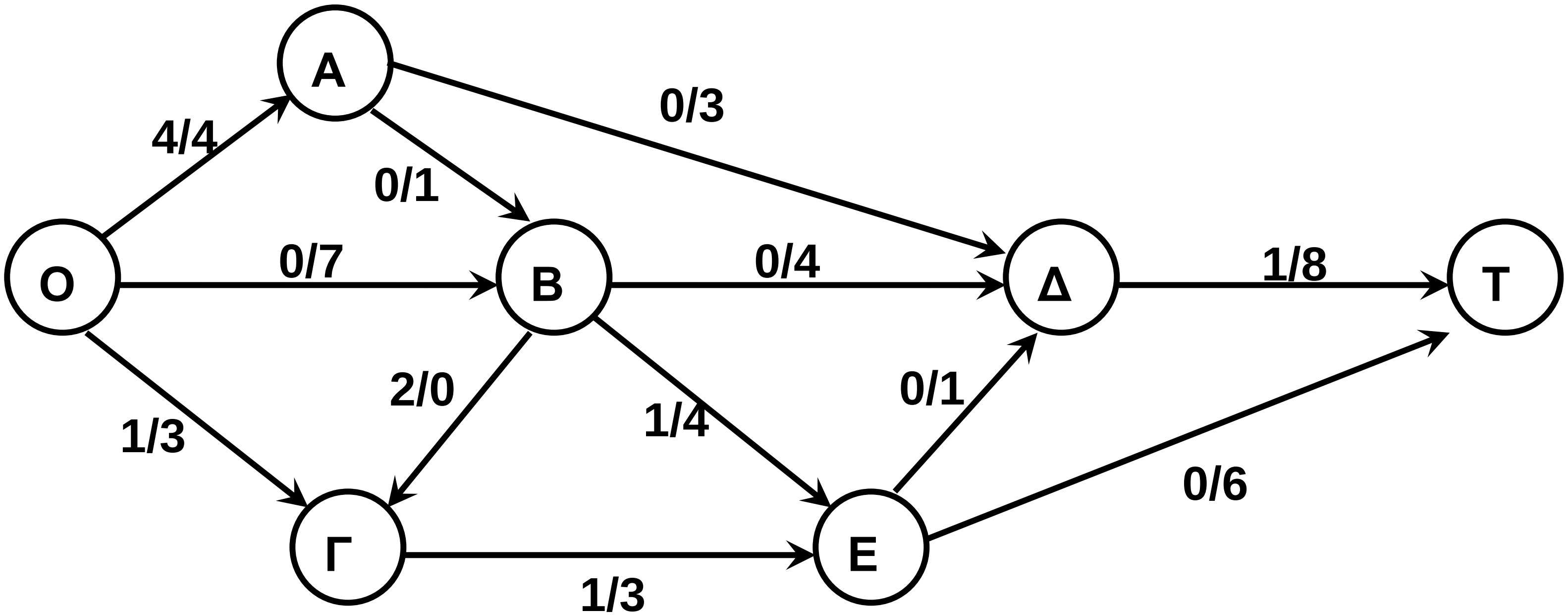
- Augmented Path ΟΓ-ΓΕ-ΕΒ-ΒΔ-ΔΤ with flow 1



- We can move along EB (against the direction of the edge) because we have already sent positive flow along BE

- Total flow 5+3+2+1+1+1+1=14

Optimal Solution



Note: Although the maximum flow will always be 14 units, there may be different combination of flows giving this result!

Maximum Flow– Minimum Cut Theorem

- **Cut**
  - **Let A a set of nodes, which includes the destination node but does not include the origin.**
  - **The set of edges (v,w) for which $v \notin A$ and $w \in A$ is called a *cut***
  - **Alternative definition: A cut is any set of edges which includes at least one edge from each path from the origin to the destination**

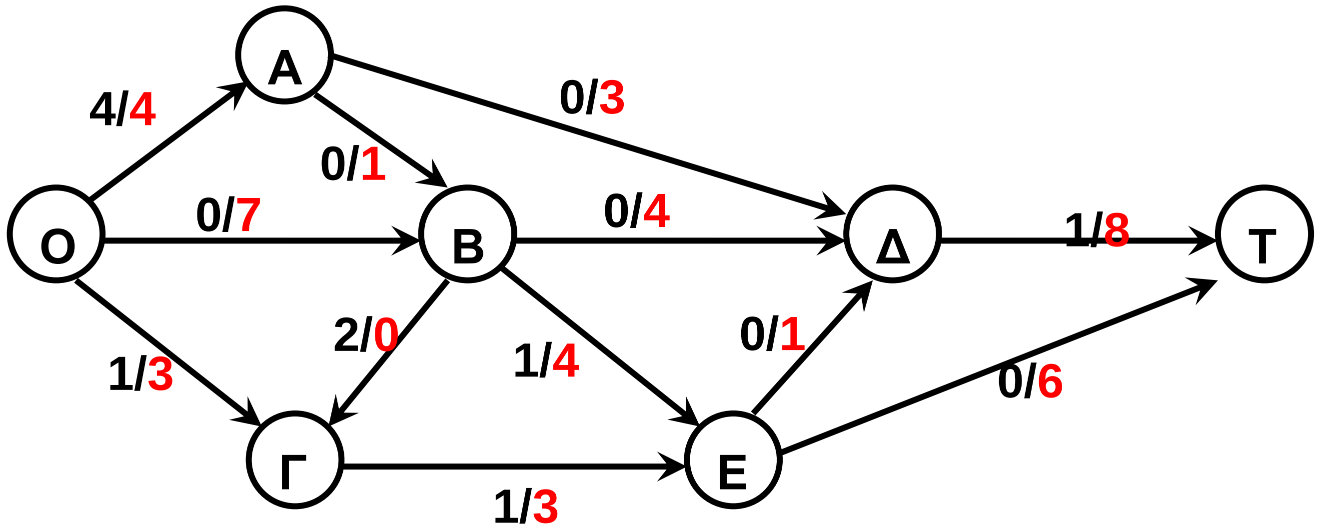
- **Practically**
  - **A cut is any set of edges which, when removed from the graph, disconnect the origin from the destination**

Maximum Flow– Minimum Cut Theorem / 2

- **Capacity of a cut**
  - **The sum of the capacities of all edges in the cut**
- **Theorem (Max Flow – Min Cut)**
  - **In a network with a node *s* as origin and a node *t* as destination, the maximum flow from *s* to *t* is equal to the minimum cut**
- **Remark**
  - **The two problems are dual to each other**
- **How to determine the Minimum Cut**
  - **Divide the nodes of the network in two subsets S_1 and S_2**
  - **S_1: all the nodes that are accessible from s following edges that are not congested yet**
  - **S_2: all other nodes**

Application in the example of slides 20-28

- **Optimal solution:**



- **Set S_1 is S_1={O, A, Γ, E, B}**
- **Set S_2 is S_2={Δ, T}**
- **The edges whose first node is in S_1 and final node in S_2 are: AΔ – EΔ – ET – BΔ**
- *(You may confirm that they are a cut. If we delete them, there is no path from O to T)*
- **The sum of their capacities is: 3+1+6+4=14 (equal to the maximum flow!)**